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# The Relation between $\tau_n$ -Tilting Modules and $n$ -term Silting Complexes

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## Abstract

In this paper, we define support  $\tau_n$ -tilting modules over a finite dimensional  $k$ -algebra  $A$ . We establish a bijection between support  $\tau_n$ -tilting modules and  $n$ -term silting complexes in the bounded homotopy category of finitely generated projective  $A$ -modules, generalizing the result of Adachi, Iyama and Reiten for support  $\tau$ -tilting modules and two-term silting complexes.

## 1 Introduction

In representation theory, one may use tilting modules and associated tilting functors to compare the module categories of two algebras  $A$  and  $B$ . The reflection functors, introduced by Bernstein, Gelfand and Ponomarev [7] in 1973 motivated tilting theory. Then Auslander, Platzeck and Reiten [6] in 1979 reformulated them and as generalisations, tilting functors were studied

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by Brenner and Butler [8]. As further generalisations, in 1982, Happel and Ringel [13] defined tilted algebras and tilting modules.

Tilting theory is one sort of generalisation of Morita equivalence. Indeed, generalised tilting modules also yield derived equivalences. In 1989 Rickard [16] proved that given two finite-dimensional algebras  $A$  and  $B$ , the bounded derived categories of  $A$  and  $B$  are equivalent if and only if there exists a tilting complex  $T$  over  $B$  such that  $A$  is the endomorphism algebra of  $T$ . Tilting complexes are generalisations of generalised tilting modules [10]. In [14] Muchtadi-Alamsyah studied the endomorphism ring of  $n$ -term tilting complexes as generalisation of 2-term tilting complexes.

Silting complexes are generalisations of tilting complexes [9, 15]. In 2014  $\tau$ -tilting theory was introduced by Adachi, Iyama and Reiten [1], thus giving a completion of tilting theory from the point-of-view of mutation, arising from the theory of cluster algebras [11, 12]. Support tilting modules (tilting modules) are generalised to support  $\tau$ -tilting modules ( $\tau$ -tilting modules) [2, 5, 17, 18]. Moreover, Adachi, Iyama and Reiten prove the following:

**Theorem** (Adachi–Iyama–Reiten [1]): *Let  $A$  be a finite-dimensional  $k$ -algebra. Then there is an explicit bijection between support  $\tau$ -tilting modules and two-term silting complexes in the bounded homotopy category of finitely generated projective  $A$ -modules  $K^b(\text{proj } A)$ .*

A natural question arises:

**Question:** *Can one generalise this result to  $n$ -term silting complexes?*

In this note, we propose a definition of support  $\tau_n$ -tilting modules and of  $n$ -term silting complexes, and we prove an explicit bijection between support  $\tau_n$ -tilting  $A$ -modules and  $n$ -term silting complexes in  $K^b(\text{proj } A)$ .

## 2 Definitions and Notations

For a finite dimensional basic algebra  $A$  over an algebraically closed field  $k$ , the category of finitely generated left  $A$ -modules is denoted by  $\text{mod } A$  and the category of finitely generated projective left  $A$ -modules is denoted by  $\text{proj } A$ . If  $M \in \text{mod } A$ ,  $\text{add } M$  is the category of all direct summands of finite direct sums of copies of  $M$ , and the number of non isomorphic indecomposable direct summands of  $M$  is denoted by  $d(M)$ .

If  $M \in \text{mod } A$  then the right  $A$ -module  $\text{Hom}_A(M, A)$  is denoted by  $M^*$ , whereas the dual, the right  $A$ -module  $\text{Hom}_k(M, k)$  is denoted by  $D(M)$ . If  $X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$  is a minimal projective presentation of  $M$ , the transpose

$Tr(M)$  of  $M$  is the cokernel of  $X_0^* \rightarrow X_1^*$ . The Auslander-Reiten translation is defined as  $\tau = DTr$ . The syzygy of  $M$ , is the kernel of  $X_0 \rightarrow M$ , and is denoted by  $\Omega(M)$ . We denote by  $\tau_n = \tau\Omega^{n-2}$ .

**Remark 2.1.** Note that  $\tau = \tau_2$  in our convention. Our notation slightly differs from the one that is currently used in higher homological algebra: what we call  $\tau_n$  here is more commonly denoted  $\tau_{n-1}$ . We chose to shift the usual notation so that  $\tau_n$ -tilting modules correspond to  $n$ -term complexes.

**Definition 2.2.** Let  $M \in \text{mod } A$ .

1. If  $\text{Hom}(M, \tau_n M) = 0$  and  $\text{Ext}^i(M, M) = 0$  for all  $i \in \{1, \dots, n-1\}$ , then we call  $M$  a **strongly  $\tau_n$ -rigid module**.
2. If  $M$  is a strongly  $\tau_n$ -rigid module and  $d(M) = d(A)$ , then we call  $M$  a  **$\tau_n$ -tilting module**.
3. If  $M$  is a  $\tau_n$ -tilting  $(A/\langle e \rangle)$ -module, where  $e$  is an idempotent of  $A$ , then we call  $M$  a **support  $\tau_n$ -tilting module**.

Let  **$s\tau_n$ -tilting** $A$  be the set of isomorphism classes of basic support  $\tau_n$ -tilting modules.

**Definition 2.3.** Let  $M \in \text{mod } A$  and  $X \in \text{proj } A$ .

1. If  $M$  is a strongly  $\tau_n$ -rigid module and  $\text{Hom}_A(X, M) = 0$ , then we call  $(M, X)$  a **strongly  $\tau_n$ -rigid pair**.
2. If  $(M, X)$  is a strongly  $\tau_n$ -rigid pair and  $d(M) + d(X) = d(A)$ , then we call  $(M, X)$  a **support  $\tau_n$ -tilting pair**.

**Proposition 2.4.** Let  $M$  be in  $\text{mod } A$  with  $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_1} X_0 \xrightarrow{d_0} M \rightarrow 0$  as its minimal projective presentation. Let  $N \in \text{mod } A$ , with  $\text{Ext}^i(M, N) = 0$  for all  $i = 1, \dots, n-1$ .

1. The following sequence

$$\begin{aligned}
 & 0 \rightarrow \text{Hom}_A(N, \tau\Omega^{n-2}M) \rightarrow D\text{Hom}_A(X_{n-1}, N) \\
 & \xrightarrow{D\text{Hom}(d_{n-1}, N)} D\text{Hom}_A(X_{n-2}, N) \xrightarrow{D\text{Hom}(d_{n-2}, N)} \dots \xrightarrow{D\text{Hom}(d_2, N)} \\
 & D\text{Hom}_A(X_1, N) \xrightarrow{D\text{Hom}(d_1, N)} D\text{Hom}_A(X_0, N) \xrightarrow{D\text{Hom}(d_0, N)} D\text{Hom}_A(M, N) \rightarrow 0
 \end{aligned}$$

is exact.

2.  $\text{Hom}_A(N, \tau\Omega^{n-2}M) = 0$  if and only if the sequence

$$\begin{aligned} \text{Hom}_A(X_0, N) &\xrightarrow{\text{Hom}(d_1, N)} \text{Hom}_A(X_1, N) \xrightarrow{\text{Hom}(d_2, N)} \dots \xrightarrow{\text{Hom}(d_{n-2}, N)} \\ &\text{Hom}_A(X_{n-2}, N) \xrightarrow{\text{Hom}(d_{n-1}, N)} \text{Hom}_A(X_{n-1}, N) \rightarrow 0 \end{aligned}$$

is exact.

*Proof.* 1. The following sequence

$$0 \rightarrow \tau\Omega^{n-2}M \rightarrow \nu X_{n-1} \rightarrow \nu X_{n-2} \cdots \rightarrow \nu X_0$$

is exact. By applying  $\text{Hom}(N, -)$ , the following diagram

$$\begin{array}{ccccccc} 0 \rightarrow \text{Hom}_A(N, \tau\Omega^{n-2}M) \rightarrow & \text{Hom}_A(N, \nu X_{n-1}) \rightarrow \dots \rightarrow & \text{Hom}_A(N, \nu X_0) & & & & \\ & \downarrow \cong & & & \downarrow \cong & & \\ & D \text{Hom}_A(X_{n-1}, N) \rightarrow \dots \rightarrow & D \text{Hom}_A(X_0, N) & \rightarrow & D \text{Hom}_A(M, N) \rightarrow 0 & & \end{array}$$

is a diagram of exact sequences.

2. It is clear from (1).

□

We denote by  $K^b(\text{proj } A)$  the homotopy category of bounded complexes of finitely generated projective  $A$ -modules and for  $\mathbf{X} \in K^b(\text{proj } A)$ , we denote by  $\text{thick } \mathbf{X}$ , the smallest full subcategory of  $K^b(\text{proj } A)$  that contains  $\mathbf{X}$  and is closed under (positive and negative) shifts, cones, isomorphisms, and direct summands.

**Definition 2.5.** Assume  $\mathbf{X} \in K^b(\text{proj } A)$ .

1. If for any  $i > 0$ ,  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{X}[i]) = 0$ , we call  $\mathbf{X}$  a **presilting complex**.
2. If  $\mathbf{X}$  is a presilting complex and also  $\text{thick } \mathbf{X} = K^b(\text{proj } A)$ , we call  $\mathbf{X}$  a **silting complex**.

The set of isomorphism classes of basic silting complexes is denoted by  $\text{silting } A$ .

**Proposition 2.6.**  $d(\mathbf{X}) = d(A)$  for any  $\mathbf{X} \in \text{silting } A$ .

*Proof* See [3, Theorem 2.27, Corollary 2.28]

**Definition 2.7.**  $\mathbf{X} = (X_i, d_i)$  in  $K^b(\text{proj } A)$  is an  $n$ -term complex if  $X_i = 0$  for all  $i \notin \{0, \dots, -(n-1)\}$  and  $H^i \mathbf{X} = 0$  for all  $i \notin \{0, -(n-1)\}$ .

Let  $n$ -presilting  $A$  (respectively,  $n$ -silting  $A$ ) be the set of isomorphism classes of basic  $n$ -term presilting (respectively, silting) complexes.

**Definition 2.8.** Let  $\mathbf{X}, \mathbf{Y}$  be objects of  $K^b(\text{proj } A)$ . We denote by  $\mathbf{X} \geq \mathbf{Y}$  if for any positive integer  $i > 0, \text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{Y}[i]) = 0$ .

**Proposition 2.9.** Let  $A$  be a finite dimensional  $k$ -algebra,  $\mathbf{X} \in K^b(\text{proj } A)$ , and  $m > 0$ . Then  $\text{length}(\mathbf{X}) \leq m$  if and only if  $A[s] \geq \mathbf{X} \geq A[m + s - 1]$  for some integer  $s$ .

*Proof.* See [4, Prop 2.9] □

**Proposition 2.10.** If  $\mathbf{X}$  is a silting complex in  $K^b(\text{proj } A)$  and  $\mathbf{Y} \in K^b(\text{proj } A)$  with  $\mathbf{X} \geq \mathbf{Y} = \mathbf{Y}_0$  then there exist triangles

$$\begin{array}{ccccccc} \mathbf{Y}_1 & \xrightarrow{g_1} & \mathbf{X}_0 & \xrightarrow{f_0} & \mathbf{Y}_0 & \rightarrow & \mathbf{Y}_1[1], \\ & & & & \vdots & & \\ \mathbf{Y}_m & \xrightarrow{g_m} & \mathbf{X}_{m-1} & \xrightarrow{f_{m-1}} & \mathbf{Y}_{m-1} & \rightarrow & \mathbf{T}_m[1], \\ & & 0 & \xrightarrow{g_{m+1}} & \mathbf{X}_1 & \xrightarrow{f_1} & \mathbf{Y}_1 \rightarrow 0 \end{array}$$

for some  $m \geq 0$  and  $f_i$  is a minimal right add  $\mathbf{X}$ -approximation.

*Proof.* See [3, Prop 2.23] □

### 3 Main Results

Let  $A$  be a finite dimensional  $k$ -algebra.

**Theorem 3.1.** There exists a bijection between

$$n\text{-silting } A \leftrightarrow s\tau_n\text{-tilting } A$$

given by  $\mathbf{X} \in n\text{-silting } A$  maps to  $H^0(\mathbf{X}) \in s\tau\text{-tilting } A$  and  $(M, Y) \in s\tau\text{-tilting } A$  maps to  $(X_{n-1} \oplus Y \xrightarrow{(d_{n-1}, 0)} X_{n-2} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_1} X_0) \in n\text{-silting } A$  where  $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_1} X_0$  is a minimal projective presentation of  $M$ .

**Proposition 3.2.** *Any object in  $n$ -presilting  $A$  is a direct summand of an object in  $n$ -silting  $A$ .*

*Proof.* Let  $\mathbf{X} \in n$ -presilting  $A$ . By Proposition 2.9, we have  $A \geq \mathbf{X} \geq A[n-1]$ . Since  $\mathbf{X} \geq A[n-1]$ , by Proposition 2.10, we have triangles

$$\begin{array}{c} \mathbf{V}_1 \xrightarrow{g_1} \mathbf{U}_0 \xrightarrow{f_0} A[n-1] \rightarrow \mathbf{V}_1[1] \\ \mathbf{V}_2 \xrightarrow{g_2} \mathbf{U}_1 \xrightarrow{f_1} \mathbf{V}_1 \rightarrow \mathbf{V}_2[1] \\ \vdots \\ \mathbf{V}_{n-1} \xrightarrow{g_{n-1}} \mathbf{U}_{n-2} \xrightarrow{f_{n-2}} \mathbf{V}_{n-2} \rightarrow \mathbf{V}_{n-1}[1] \end{array}$$

with  $f_i$  minimal right add  $\mathbf{P}$ -approximation. Put  $\mathbf{W} = \mathbf{X} \oplus \mathbf{V}_{n-1}$ . Since  $A$  is a silting object, then  $K^b(\text{proj } A) = \text{thick } \mathbf{W}$ .

1. We prove that for any  $i > 0$ ,  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{V}_{n-1}[i]) = 0$ . As there exists an exact sequence

$$\begin{aligned} \text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{V}_{n-2}[i-1]) &\xrightarrow{0} \text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{V}_{n-1}[i]) \\ &\rightarrow \text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{U}_{n-2}[i]) = 0. \end{aligned}$$

We observe that for any  $i > 0$ ,  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{V}_{n-1}[i]) = 0$ .

2. We will prove that for any  $i > 0$ ,  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{V}_{n-1}, \mathbf{W}[i]) = 0$ . As the following sequences

$$0 \stackrel{(1)}{=} \text{Hom}_{K^b(\text{proj } A)}(\mathbf{U}_{n-2}, \mathbf{W}[i]) \rightarrow \text{Hom}_{K^b(\text{proj } A)}(\mathbf{V}_{n-1}, \mathbf{W}[i]) \rightarrow$$

$$\text{Hom}_{K^b(\text{proj } A)}(\mathbf{V}_{n-2}[-1], \mathbf{W}[i]) \rightarrow \text{Hom}_{K^b(\text{proj } A)}(\mathbf{U}_{n-2}[-1], \mathbf{W}[i]) \stackrel{(1)}{=} 0,$$

are exact, therefore  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{V}_{n-1}, \mathbf{W}[i]) \cong \text{Hom}_{K^b(\text{proj } A)}(\mathbf{V}_{n-2}[-1], \mathbf{W}[i])$ .

There are exact sequences :

$$0 = \text{Hom}(\mathbf{U}_{n-3}[-1], \mathbf{W}[i]) \rightarrow \text{Hom}(\mathbf{V}_{n-2}[-1], \mathbf{W}[i]) \rightarrow$$

$$\text{Hom}(\mathbf{V}_{n-3}[-2], \mathbf{W}[i]) \rightarrow \text{Hom}(\mathbf{U}_{n-3}[-2], \mathbf{W}[i]) = 0,$$

$$0 = \text{Hom}(\mathbf{U}_{n-4}[-2], \mathbf{W}[i]) \rightarrow \text{Hom}(\mathbf{V}_{n-3}[-2], \mathbf{W}[i]) \rightarrow$$

$$\text{Hom}(\mathbf{V}_{n-4}[-3], \mathbf{W}[i]) \rightarrow \text{Hom}(\mathbf{U}_{n-4}[-3], \mathbf{W}[i]) = 0,$$

⋮

$$0 = \text{Hom}(\mathbf{U}_0[2-n], \mathbf{W}[i]) \rightarrow \text{Hom}(\mathbf{V}_1[2-n], \mathbf{W}[i]) \rightarrow \text{Hom}(A, \mathbf{W}[i]) \rightarrow \text{Hom}(\mathbf{U}_0[1-n], \mathbf{W}[i]) = 0.$$

Since  $A \geq \mathbf{X}$ , we have  $\text{Hom}_{K^b(\text{proj } A)}(A, \mathbf{X}[i]) = 0$  and there are exact sequences

$$0 = \text{Hom}(A, \mathbf{U}_{n-2}[i]) \rightarrow \text{Hom}(A, \mathbf{V}_{n-2}[i]) \rightarrow \text{Hom}(A, \mathbf{V}_{n-1}[i+1]) \rightarrow \text{Hom}(A, \mathbf{U}_{n-2}[i+1]) = 0,$$

$$0 = \text{Hom}(A, \mathbf{U}_{n-3}[i]) \rightarrow \text{Hom}(A, \mathbf{V}_{n-3}[i]) \rightarrow \text{Hom}(A, \mathbf{V}_{n-2}[i+1]) \rightarrow \text{Hom}(A, \mathbf{U}_{n-3}[i+1]) = 0,$$

$\vdots$

$$0 = \text{Hom}(A, \mathbf{U}_0[i]) \rightarrow \text{Hom}(A, A[n-1+i]) \rightarrow \text{Hom}(A, \mathbf{V}_1[i+1]) \rightarrow \text{Hom}(A, \mathbf{U}_0[i+1]) = 0.$$

Finally, we get

$$\begin{aligned} \text{Hom}_{K^b(\text{proj } A)}(\mathbf{V}_{n-1}, \mathbf{W}[i]) &\cong \text{Hom}_{K^b(\text{proj } A)}(\mathbf{V}_{n-2}[-1], \mathbf{W}[i]) \\ &\cong \text{Hom}_{K^b(\text{proj } A)}(A, \mathbf{W}[i]) \\ &= \text{Hom}_{K^b(\text{proj } A)}(A, \mathbf{X}[i]) \oplus \text{Hom}_{K^b(\text{proj } A)}(A, \mathbf{V}_{n-2}[i]) \\ &\cong 0 \oplus \text{Hom}_{K^b(\text{proj } A)}(A, A[n-1+i]) = 0. \end{aligned}$$

By (1) and (2), we see that for any  $i > 0$ ,  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{W}, \mathbf{W}[i]) = 0$ . □

**Proposition 3.3.** *If  $\mathbf{X} \in n$ -presilting  $A$  then  $\mathbf{X} \in n$ -silting  $A$  if and only if  $d(\mathbf{X}) = d(A)$ .*

*Proof.* According to Proposition 2.6,  $d(\mathbf{X}) = d(A)$ . We will show the other direction. Let  $\mathbf{X} \in n$ -presilting  $A$  with  $d(\mathbf{X}) = d(A)$ . By Proposition 3.2,  $\mathbf{X} \oplus \mathbf{Y}$  is silting for some complex  $\mathbf{Y}$ . Therefore by Proposition 2.6,  $d(\mathbf{X} \oplus \mathbf{Y}) = d(A) = d(\mathbf{X})$ . This implies  $\mathbf{Y}$  is in  $\text{add } \mathbf{X}$  and  $\mathbf{X}$  is silting. □

**Lemma 3.4.** *Let  $M, N$  in  $\text{mod } A$ . Let  $X_{n-1} \xrightarrow{d_{n-1}^X} X_{n-2} \xrightarrow{d_{n-2}^X} \dots \xrightarrow{d_1^X} X_0 \xrightarrow{d_0^X} M \rightarrow 0$  be a minimal projective presentation of  $M$  and  $Y_{n-1} \xrightarrow{d_{n-1}^Y} Y_{n-2} \xrightarrow{d_{n-2}^Y} \dots \xrightarrow{d_1^Y} Y_0 \xrightarrow{d_0^Y} N \rightarrow 0$  be minimal projective presentation of  $N$ . Denote by  $\mathbf{X} = (X_{n-1} \xrightarrow{d_{n-1}^X} X_{n-2} \xrightarrow{d_{n-2}^X} \dots \xrightarrow{d_1^X} X_0)$  and  $\mathbf{Y} = (Y_{n-1} \xrightarrow{d_{n-1}^Y} Y_{n-2} \xrightarrow{d_{n-2}^Y} \dots \xrightarrow{d_1^Y} Y_0)$   $n$ -term complexes over  $A$ . Then the following assertions are equivalent:*

1.  $\text{Hom}_A(N, \tau_n M) = 0$  and for all  $i = 1, \dots, n-1$ ,  $\text{Ext}^i(M, N) = 0$ .
2.  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{Y}[i]) = 0$  for all  $i = 1, \dots, n-1$ .

*This implies that  $M$  is strongly  $\tau_n$ -rigid if and only if  $\mathbf{X} \in n$ -presilting  $A$ .*



*Proof.* (1  $\Rightarrow$  2) Let  $i \in \{1, \dots, n-1\}$ . Any morphism  $f \in \text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{Y}[i])$  is defined by some  $f_j \in \text{Hom}_A(X_j, Y_{j-i})$ , for all  $j \in \{1, \dots, n-1\}$ .

$$\begin{array}{ccccccccccc} & & 0 & \rightarrow & X_{n-1} & \xrightarrow{d_{n-1}^X} & \cdots & \rightarrow & X_i & \xrightarrow{d_i^X} & X_{i-1} & \xrightarrow{d_{i-1}^X} & \cdots & \rightarrow & X_0 & \rightarrow & 0 \\ & & \downarrow & & \downarrow f_{n-1} & & & & \downarrow f_i & & \downarrow & & & & & & & \\ 0 & \rightarrow & Y_{n-1} & \xrightarrow{d_{n-1}^Y} & \cdots & \rightarrow & Y_{n-i} & \xrightarrow{d_{n-i}^Y} & Y_{n-1-i} & \xrightarrow{d_{n-1-i}^Y} & \cdots & \rightarrow & Y_0 & \rightarrow & 0 & & \end{array}$$

By Proposition 2.4, since  $\text{Hom}_A(N, \tau_n M) = 0$ ,

$$\text{Hom}_A(X_0, N) \xrightarrow{(d_1^X, N)} \text{Hom}_A(X_1, N) \xrightarrow{(d_2^X, N)} \cdots \xrightarrow{(d_{n-2}^X, N)} \text{Hom}_A(X_{n-2}, N) \xrightarrow{(d_{n-1}^X, N)} \text{Hom}_A(X_{n-1}, N) \rightarrow 0 \text{ is exact.}$$

As  $\text{Ext}^{i+1}(M, N) = 0$ , then  $(d_i^X, N) : \text{Hom}_A(X_{i-1}, N) \rightarrow \text{Hom}_A(X_i, N)$  is surjective. Therefore there exists  $f_{i-1} : X_{i-1} \rightarrow N$  such that  $d_0^Y f_i = f_{i-1} d_i^X$ . Moreover, since  $X_{i-1}$  is projective, there exists  $h_{i-1} : X_{i-1} \rightarrow Y_0$  such that  $d_0^Y h_{i-1} = f_{i-1}$ . Since  $d_0^Y(f_i - h_{i-1} d_i^X) = 0$ , we have  $h_i : X_i \rightarrow Y_1$  with  $f_i = d_1^Y h_i + h_{i-1} d_i^X$ .

$$\begin{array}{ccccccccccccccc} & & 0 & \rightarrow & X_{n-1} & \xrightarrow{d_{n-1}^X} & \cdots & \rightarrow & X_i & \xrightarrow{d_i^X} & X_{i-1} & \xrightarrow{d_{i-1}^X} & X_{i-2} & \rightarrow & \cdots & \rightarrow & X_0 & \rightarrow & M & \rightarrow & 0 \\ & & \downarrow & & \downarrow f_{n-1} & & & & \downarrow f_i & & \downarrow f_{i-1} & & \downarrow & & & & & & & & & & \\ 0 & \rightarrow & Y_{n-1} & \rightarrow & \cdots & \rightarrow & Y_{n-i} & \xrightarrow{d_{n-i}^Y} & Y_{n-1-i} & \xrightarrow{d_{n-1-i}^Y} & \cdots & \rightarrow & Y_0 & \xrightarrow{d_0^Y} & N & \rightarrow & 0 & & & & & & \end{array}$$

Now since  $f_i d_{i-1}^X = d_1^Y f_{i-1}$ , then  $d_1^Y(f_{i-1} - h_i d_{i-1}^X) = 0$ , and we have  $h_{i+1} : X_{i+1} \rightarrow Y_2$  with  $f_{i+1} = d_2^Y h_{i+1} + h_i d_{i+1}^X$ . We continue recursively and get that  $f_j = d_{j-i+1}^Y h_j + h_{j-1} d_j^X$  for all  $j$  between  $i+2$  and  $n-1$ . Hence for all  $i > 0$ ,  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{Y}[i]) = 0$ .

(2  $\Rightarrow$  1) Take any  $f \in \text{Hom}_A(X_{n-1}, N)$ . Then since  $X_{n-1}$  is projective,  $d_0^Y g = f$  for some  $g : X_{n-1} \rightarrow Y_0$ .

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{d_{n-1}^X} & X_{n-2} \\ \downarrow g & \searrow f & \\ Y_1 & \rightarrow & Y_0 \xrightarrow{d_0^Y} N \rightarrow 0 \end{array}$$

Hence  $g$  induces a morphism  $\mathbf{X} \rightarrow \mathbf{Y}[n-1]$  in  $K^b(\text{proj } A)$ .

As  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{Y}[n-1]) = 0$ , there exists  $h_{n-2} : X_{n-2} \rightarrow Y_0$  and

$h_{n-1} : X_{n-1} \rightarrow Y_1$  such that  $g = d_1^Y h_{n-1} + h_{n-2} d_{n-1}^X$ . Hence we have  $f = d_0^Y (d_1^Y h_{n-1} + h_{n-2} d_{n-1}^X) = d_0^Y h_{n-2} d_{n-1}^X$ . Therefore  $(d_{n-1}^X, N)$  is surjective and by Proposition 2.4, we conclude that  $\text{Hom}_A(N, \tau_n M) = 0$ .

Now let  $i \in \{1, \dots, n-1\}$ . It is clear that  $\text{Ext}^i(M, N) = \text{Ext}^1(\Omega^{i-1} M, N)$  and this is zero if  $\text{Hom}_A(X_{i-1}, N) \rightarrow \text{Hom}_A(\Omega^i M, N)$  is surjective. Take any  $f_i \in \text{Hom}_A(\Omega^i M, N)$ . This homomorphism induces  $f'_i : X_i \rightarrow N$ . Since  $X_i$  is projective, there exists  $g_i : X_i \rightarrow Q_0$  such that  $d_0^Y g_i = f'_i$ .

$$\begin{array}{ccc} X_i & \xrightarrow{d_i^X} & X_{i-1} \\ \downarrow g_{i-1} & \searrow f'_i & \\ Y_1 \rightarrow Y_0 & \xrightarrow{d_0^Y} & N \rightarrow 0 \end{array}$$

We see that  $g_i$  induces a morphism  $\mathbf{X} \rightarrow \mathbf{Y}[i]$  in  $K^b(\text{proj } A)$ . As  $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{X}, \mathbf{Y}[i]) = 0$ , there exists  $h_{i-1} : X_{i-1} \rightarrow Y_0$  and  $h_i : X_i \rightarrow Y_1$  such that  $g_i = d_1^Y h_i + h_{i-1} d_i^X$ . Hence we have  $f'_i = d_0^Y (d_1^Y h_i + h_{i-1} d_i^X) = d_0^Y h_{i-1} d_i^X$  and  $f_i = d_0^Y h_{i-1} s_i$  where  $s_i$  is the inclusion  $\Omega^i M \rightarrow X_{i-1}$ . Therefore  $(s_i, N)$  surjective and  $\text{Ext}^i(M, N) = 0$ . □

**Lemma 3.5.** *Let  $M \in \text{mod } A$ . Let  $\mathbf{X} := (X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0)$  be an  $n$ -term complex where  $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} M \rightarrow 0$  is a minimal projective presentation of  $M$ . Let  $Y \in \text{proj } A$ . Then*

$$\text{Hom}_A(Y, M) = 0 \text{ if and only if } \text{Hom}_{K^b(\text{proj } A)}(Y, \mathbf{X}) = 0.$$

By the following result we will see that  $n$ -term silting complexes define support  $\tau_n$ -tilting modules.

**Proposition 3.6.** *Let  $\mathbf{X} := (X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0)$  be an  $n$ -term complex for  $A$ . Let  $M = \text{Cok } d_1$ .*

1. *If  $\mathbf{X}$  is a silting complex for  $A$  where  $d_{n-1}$  is a right minimal homomorphism, then  $M$  is a  $\tau_n$ -tilting module.*
2. *If  $\mathbf{X}$  is a silting complex for  $A$  then  $M$  is a support  $\tau_n$ -tilting module.*

*Proof.* 1. Denote by  $d_{n-1} = (d'_{n-1} \ 0) : X'_{n-1} \oplus X''_{n-1} \rightarrow X_{n-2}$ , where  $d'_{n-1}$  is right minimal. Then  $X'_{n-1} \xrightarrow{d_{n-1}'} X_{n-2} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_1} X_1 \xrightarrow{p_1}$

$X_0 \xrightarrow{d_0} M \rightarrow 0$  is a minimal projective presentation of  $M$ . We will prove that  $(M, X''_{n-1})$  is a support  $\tau_n$ -tilting pair. As  $\mathbf{X}$  is a silting complex, then  $\text{Hom}_{K^b(\text{proj } A)}(X''_{n-1}, \mathbf{X}) = 0$ , and by Lema 3.5 we obtain  $\text{Hom}_A(X''_{n-1}, M) = 0$ . Therefore  $(M, X''_{n-1})$  is a strongly  $\tau_n$ -rigid pair. Now we obtain

$$d(M) = d(X'_{n-1} \xrightarrow{d'_{n-1}} X_{n-2} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0),$$

because  $d'_{n-1}$  is a minimal projective presentation of  $M$ . Thus

$$d(M) + d(X''_{n-1}) = d(X'_{n-1} \xrightarrow{d'_{n-1}} X_{n-2} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0) + d(P''_{n-1}) = d(P),$$

and by Proposition 2.6 this is equal to  $d(A)$ . We conclude the assertion.

2. For  $X''_{n-1} = 0$  in (1), we get the assertion.

□

We will see that support  $\tau_n$ -tilting modules define  $n$ -term silting complexes for  $A$ .

**Proposition 3.7.** *Let  $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0 \rightarrow M \rightarrow 0$  be a minimal projective presentation for  $M$ .*

1. *If  $M$  is  $\tau_n$ -tilting, then  $(X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0)$  is a  $n$ -term silting complex.*
2. *If  $(M, Y)$  is a support  $\tau_n$ -tilting pair then  $(X_{n-1} \oplus Y \xrightarrow{(d_{n-1} \ 0)} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0)$  is a  $n$ -term silting complex.*

*Proof.* 2. By Lemma 3.4,  $\mathbf{X} := (X_{n-1} \oplus Y \xrightarrow{(d_{n-1} \ 0)} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0)$  is a presilting complex. As  $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0 \rightarrow M \rightarrow 0$  is a minimal projective presentation, we have  $d(M) + d(Y) = d(A)$ . Thus we have

$$\begin{aligned} d(\mathbf{X}) &= d(X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0) + d(Y) \\ &= d(M) + d(Y) = d(A). \end{aligned}$$

Therefore  $\mathbf{X}$  is silting by Proposition 2.6.

1. For  $Q = 0$  in 2 we obtain the assertion. □

The proof of Theorem 3.1 is obtained by using Proposition 3.6 and Proposition 3.7.

## 4 Further Research

When the algebra  $A$  is cluster-tilted (or more generally 2-Calabi–Yau tilted), there is a functor from a cluster category (or 2-Calabi–Yau triangulated category)  $\mathcal{C}$  to  $\text{mod } A$ . The support  $\tau$ -tilting modules over  $A$  are precisely the images of the cluster-tilting objects of  $\mathcal{C}$  under this functor. It seems thus natural to consider  $n$ -Calabi–Yau tilted algebras and try to establish a bijection between  $n$ -cluster tilting objects and support  $\tau_n$  tilting modules. Once the bijection is established, one may study mutations and geometric realisation of  $\tau_n$ -tilting modules and  $n$ -term silting complexes.

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