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# Monotonicity in half-spaces of positive solutions to $-\Delta_p u = f(u)$ in the case $p > 2$

ALBERTO FARINA, LUIGI MONTORO AND BERARDINO SCIUNZI

**Abstract.** We consider weak distributional solutions to the equation  $-\Delta_p u = f(u)$  in half-spaces under zero Dirichlet boundary condition. We assume that the nonlinearity is positive and superlinear at zero. For  $p > 2$  (the case  $1 < p \leq 2$  is already known) we prove that any positive solution is strictly monotone increasing in the direction orthogonal to the boundary of the half-space. As a consequence we deduce some Liouville-type theorems for the Lane-Emden-type equation. Furthermore any nonnegative solution turns out to be  $C^{2,\alpha}$  smooth.

**Mathematics Subject Classification (2010):** 35B05 (primary); 35B65, 35J70 (secondary).

## 1. Introduction

We consider the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \mathbb{R}_+^N \\ u(x', y) \geq 0 & \text{in } \mathbb{R}_+^N \\ u(x', 0) = 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases} \quad (1.1)$$

where  $N \geq 2$  and  $f(\cdot)$  satisfies:

$(h_f)$  the nonlinearity  $f$  is *positive*, i.e.,  $f(t) > 0$  for  $t > 0$ , it is locally Lipschitz continuous in  $\mathbb{R}^+ \cup \{0\}$ , and

$$\lim_{t \rightarrow 0^+} \frac{f(t)}{t^{p-1}} = f_0 \in \mathbb{R}^+ \cup \{0\}.$$

In the following we denote a generic point in  $\mathbb{R}^N$  by  $(x', y)$  with  $x' = (x_1, x_2, \dots, x_{N-1})$  and  $y = x_N$ , we assume with no loss of generality that  $\mathbb{R}_+^N = \{y > 0\}$ . Furthermore, according to the regularity results in [18, 32, 41] (see also the recent

developments in [31,40]), we assume that  $u \in C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{R}_+^N})$  and that it fulfills the equation in the weak distributional sense. Actually, in our case the regularity up to the boundary does not follow directly from [32] and an argument by reflection is needed. This is quite standard and will be described later on in this paper.

By the strong maximum principle (see [42]), it follows that any nonnegative nontrivial solution is actually (strictly) positive. In this case we study the monotonicity of the solution in the direction orthogonal to the boundary of the half-space.

The main tool is the Alexandrov-Serrin moving plane method that dates back to [1,39]. It is well known that the moving plane procedure allows one to prove monotonicity and symmetry properties of the solutions to general PDE's. In the case of bounded domains and in the semilinear case  $p = 2$ , this study was started in the celebrated papers [5,27]. In the case of unbounded domains the main examples, arising from many applications, are provided by the whole space  $\mathbb{R}^N$  and by the half-space  $\mathbb{R}_+^N$ . We refer to [7,27,28] for the case of the whole space, where radial symmetry of the solutions is expected. In this paper we will address the case when the domain is a half-space. We refer the reader to [2–4,10,16,17,19,25,35] for previous results concerning monotonicity of the solutions in half-spaces, in the non-degenerate case.

The case of  $p$ -Laplace equations is really harder to study. In fact the  $p$ -Laplacian is a nonlinear operator and, as a consequence, comparison principles are not equivalent to maximum principles. The degenerate nature of the operator also causes lack of regularity of the solutions. Furthermore, in the case  $p > 2$  that we are considering, the use of weighted Sobolev spaces is naturally associated to the study of qualitative properties of the solutions. This issue is more delicate in unbounded domains. We cannot describe here in full detail, this fact, that will be clarified in the body of the paper. Let us only say that the use of weighted Sobolev spaces is necessary in the case  $p > 2$ , and it requires in turn the use of a weighted Poincaré type inequality with weight  $\rho = |\nabla u|^{p-2}$  (see [13]). The latter involves constants that may blow up when the solution approaches zero, and thus may happen also for positive solutions in unbounded domains. Once again the lack of compactness plays an important role.

The first results in bounded domains and in the case  $1 < p < 2$  were obtained in [12]. The case  $p > 2$  requires the above-mentioned use of weighted Sobolev spaces and was solved in [13], for positive nonlinearities ( $f(t) > 0$  for  $t > 0$ ). In the case of the whole space, we refer the reader to the recent results in [11,38,43].

The first results concerning the  $p$ -Laplace operator and problems in half-spaces have been obtained in [15] in dimension two. The same techniques have been also exploited in the fully nonlinear case in [8]. In higher dimension, the first results have been obtained in the singular case  $1 < p < 2$  in [21,23] (see also [26]) where positive locally Lipschitz continuous nonlinearities are considered. A partial answer in the more difficult degenerate case  $p > 2$  was obtained in [22], where power-like nonlinearities are considered under the restriction  $2 < p < 3$ . Here, considering a larger class of nonlinearities, namely positive nonlinearities that are superlinear at zero, we remove the condition  $2 < p < 3$  and prove the following:

**Theorem 1.1.** *Let  $p > 2$  and let  $u \in C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{R}_+^N})$  be a positive solution to (1.1) with  $|\nabla u| \in L^\infty(\mathbb{R}_+^N)$ . Then, under the assumption  $(h_f)$ , it follows that*

$$\frac{\partial u}{\partial y} > 0 \quad \text{in } \mathbb{R}_+^N.$$

*As a consequence  $u \in C_{\text{loc}}^{2,\alpha'}(\overline{\mathbb{R}_+^N})$  for some  $0 < \alpha' < 1$ .*

Our monotonicity result holds in particular for Lane-Emden type equations, namely in the case  $f(u) = u^q$  with  $q \geq p - 1$ . Note that, the case  $q \leq p - 1$ , or more generally the case when, for some  $t_0 > 0$ , it holds

$$f(t) \geq c t^{p-1} \quad \text{for } t \in [0, t_0],$$

is already contained in [22, Theorem 3]. Furthermore Theorem 1.1 is proved without *a-priori* assumptions on the behavior of the solution, that is, at infinity the solution may decay at zero in some regions, while it can be far from zero in some other regions. It is crucial that, in our result, only local regularity of the solution is required. Indeed assuming that the solution has summability properties at infinity, *i.e.*, assuming that the solution belongs to some Sobolev space, the monotonicity result is somehow easier to deduce and it generally leads to the nonexistence of such solutions (we refer to [33], see also [44]). Finally it is worth emphasizing that we prove the first step of the moving plane procedure in a very general setting. Indeed, in Theorem 3.1, we prove that any positive solution is monotone increasing near the boundary for any  $1 < p < \infty$  only assuming that the nonlinearity  $f$  is continuous in  $\mathbb{R}^+ \cup \{0\}$  and for some  $T > 0$ , it holds that  $|f(t)| \leq \bar{k} t^{p-1}$  for  $t \in [0, T]$  and some  $\bar{k} = \bar{k}(T) > 0$ .

The technique developed to prove Theorem 1.1 also allows us to deduce a monotonicity result for solutions to equations involving a different class of nonlinearities. We have the following

**Theorem 1.2.** *Let  $p > 2$  and let  $u \in C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{R}_+^N}) \cap W^{1,\infty}(\mathbb{R}_+^N)$  be a positive solution to (1.1). Suppose that  $f(\cdot)$  is locally Lipschitz continuous in  $\mathbb{R}^+ \cup \{0\}$  and that there exists  $t_0 > 0$  such that*

$$f(s) > 0 \quad \text{for } 0 < t < t_0, \quad f(s) < 0 \quad \text{for } t > t_0.$$

*Assume furthermore that*

$$\lim_{t \rightarrow 0^+} \frac{f(t)}{t^{p-1}} = f_0 \in \mathbb{R}^+ \cup \{0\}, \quad \lim_{t \rightarrow t_0} \frac{f(t)}{(t_0 - t)|t_0 - t|^{p-2}} = f^0 \in \mathbb{R}^+ \cup \{0\}. \quad (1.2)$$

*Then*

$$\frac{\partial u}{\partial y} > 0 \quad \text{in } \mathbb{R}_+^N.$$

*As a consequence,  $u \in C_{\text{loc}}^{2,\alpha'}(\overline{\mathbb{R}_+^N})$  for some  $0 < \alpha' < 1$ .*

Theorem 1.2 is mainly a corollary of Theorem 1.1 and it extends to the degenerate case  $p > 2$  earlier results in [23] (see Theorem 1.3 there and see also [21, Theorem 1.8]). It applies, for instance, to solutions of

$$-\Delta_p u = u(1 - u^2)|1 - u^2|^q,$$

where  $q \geq p - 2$ . When  $p = 2$  and  $q = 0$ , the above equation reduces to

$$-\Delta u = u(1 - u^2),$$

which is the celebrated Allen-Cahn equation arising in a famous conjecture of De Giorgi.

The monotonicity of the solution implies in particular stability its, see [9, 24]. This allows us to deduce some Liouville-type theorems. Following [9, 20], we set

$$q_c(N, p) = \frac{[(p-1)N - p]^2 + p^2(p-2) - p^2(p-1)N + 2p^2\sqrt{(p-1)(N-1)}}{(N-p)[(p-1)N - p(p+3)]}.$$

We refer to [9, 20] and the references therein for more details and we only note here that the exponent  $q_c(N, p)$  is larger than the classical critical Sobolev exponent. Once that, we know that by Theorem 1.1, the solutions are monotone and therefore stable, the same proof of [22, Theorem 4] provides the following Liouville-type result:

**Theorem 1.3.** *Let  $p > 2$  and let  $u \in C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{R}_+^N})$  be a non-negative weak solution of (1.1) in  $\mathbb{R}_+^N$  with  $|\nabla u| \in L^\infty(\mathbb{R}_+^N)$  and*

$$f(u) = u^q.$$

*Assume that*

$$\begin{cases} (p-1) < q < \infty & \text{if } N \leq \frac{p(p+3)}{p-1} \\ (p-1) < q < q_c(N, p) & \text{if } N > \frac{p(p+3)}{p-1}. \end{cases}$$

*Then  $u = 0$ . If moreover we assume that  $u$  is bounded, then it follows that  $u = 0$  assuming only that*

$$\begin{cases} (p-1) < q < \infty & \text{if } (N-1) \leq \frac{p(p+3)}{p-1} \\ (p-1) < q < q_c((N-1), p) & \text{if } (N-1) > \frac{p(p+3)}{p-1}. \end{cases}$$

The paper is organized as follows. In Section 2 we recall some known results for the reader's convenience. In Section 3 we prove some preliminary results and then we prove Theorems 1.1 and 1.2.

## 2. Preliminaries

We start by stating some notation and preliminary results. Generic fixed and numerical constants will be denoted by  $C$  (with subscript in some case) and they will be allowed to vary within a single line or formula.

For  $0 \leq \alpha < \beta$ , we define the strip  $\Sigma_{(\alpha, \beta)}$  as

$$\Sigma_{(\alpha, \beta)} := \mathbb{R}^{N-1} \times (\alpha, \beta) \quad (2.1)$$

and we will denote by  $\Sigma_\beta$  the strip

$$\Sigma_\beta := \mathbb{R}^{N-1} \times (0, \beta).$$

Then we define the cylinder

$$\mathcal{C}_{(\alpha, \beta)}(R) = \mathcal{C}(R) := \Sigma_{(\alpha, \beta)} \cap \{B'(0, R) \times \mathbb{R}\}, \quad (2.2)$$

where  $B'(0, R)$  is the ball in  $\mathbb{R}^{N-1}$  of radius  $R$  and center at zero. Given  $\lambda \in \mathbb{R}$  we will define  $u_\lambda(x)$  by

$$u_\lambda(x) = u_\lambda(x', y) := u(x', 2\lambda - y) \quad \text{in } \Sigma_{2\lambda}. \quad (2.3)$$

Finally we use the notation

$$u^+ := \max\{u, 0\}.$$

In the sequel of the paper we will often use the strong maximum principle. We refer to [42] (see also [34]) and we recall here the statement.

**Theorem 2.1 (Strong maximum principle and Hopf's lemma).** *Let  $\Omega$  be a domain in  $\mathbb{R}^N$  and suppose that  $u \in C^1(\Omega)$ ,  $u \geq 0$  in  $\Omega$ , weakly solves*

$$-\Delta_p u + cu^q = g \geq 0 \quad \text{in } \Omega,$$

*with  $1 < p < \infty$ ,  $q \geq p - 1$ ,  $c \geq 0$  and  $g \in L_{\text{loc}}^\infty(\Omega)$ . If  $u \neq 0$  then  $u > 0$  in  $\Omega$ . Moreover for any point  $x_0 \in \partial\Omega$  where the interior sphere condition is satisfied, and such that  $u \in C^1(\Omega \cup \{x_0\})$  and  $u(x_0) = 0$  we have that  $\frac{\partial u}{\partial \nu} > 0$  for any inward directional derivative, that is, if  $y$  approaches  $x_0$  in a ball  $B \subseteq \Omega$  that has  $x_0$  on its boundary, then  $\lim_{y \rightarrow x_0} \frac{u(y) - u(x_0)}{|y - x_0|} > 0$ .*

Let us recall that the linearized operator  $L_u(v, \varphi)$  for a fixed solution  $u$  of  $-\Delta_p(u) = f(u)$  is well defined for every  $v, \varphi \in H_\rho^{1,2}(\Omega)$  with  $\rho \equiv |\nabla u|^{p-2}$ , by

$$L_u(v, \varphi) \equiv \int_\Omega [|\nabla u|^{p-2}(\nabla v, \nabla \varphi) + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla v)(\nabla u, \nabla \varphi) - f'(u)v\varphi].$$

We refer [13] for more details and in particular for the definition of the weighted Sobolev spaces involved. Let us only recall here that the space  $H_{\rho}^{1,2}(\Omega)$  can be defined as the space of functions  $v$  such that  $\|v\|_{H_{\rho}^{1,2}(\Omega)}$  is bounded and

$$\|v\|_{H_{\rho}^{1,2}(\Omega)} := \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega, \rho)}.$$

This is the same space obtained performing the completion of smooth functions under the norm above. The space  $H_{0,\rho}^{1,2}(\Omega)$  is obtained taking the closure of  $C_c^{\infty}(\Omega)$  under the same norm and  $\|\nabla v\|_{L^2(\Omega, \rho)}$  is an equivalent norm in  $H_{0,\rho}^{1,2}(\Omega)$ .

Moreover,  $v \in H_{\rho}^{1,2}(\Omega)$  is a weak solution of the linearized equation if

$$L_u(v, \varphi) = 0$$

for any  $\varphi \in H_{0,\rho}^{1,2}(\Omega)$ . By [13] we have that  $u_{x_i} \in H_{\rho}^{1,2}(\Omega)$  for  $i = 1, \dots, N$ , and  $L_u(u_{x_i}, \varphi)$  is well defined for every  $\varphi \in H_{0,\rho}^{1,2}(\Omega)$ , with

$$L_u(u_{x_i}, \varphi) = 0 \quad \forall \varphi \in H_{0,\rho}^{1,2}(\Omega).$$

In other words, the derivatives of  $u$  are weak solutions of the linearized equation. Consequently by the strong maximum principle for the linearized operator (see [14]) we have the following

**Theorem 2.2.** *Let  $u \in C^1(\overline{\Omega})$  be a weak solution of  $-\Delta_p(u) = f(u)$  in a bounded smooth domain  $\Omega$  of  $\mathbb{R}^N$  with  $\frac{2N+2}{N+2} < p < \infty$ ,  $f$  positive ( $f(s) > 0$  for  $s > 0$ ) and locally Lipschitz continuous. Then, for any  $i \in \{1, \dots, N\}$  and any domain  $\Omega' \subset \Omega$  with  $u_{x_i} \geq 0$  in  $\Omega'$ , we have that either  $u_{x_i} \equiv 0$  in  $\Omega'$  or  $u_{x_i} > 0$  in  $\Omega'$ .*

We now state the weighted Poincaré-type inequality proved in [13] that will be useful in the sequel.

**Theorem 2.3 (Weighted Poincaré-type inequality).** *Let  $w \in H_{\rho}^{1,2}(\Omega)$  be such that*

$$|w(x)| \leq \hat{C} \int_{\Omega} \frac{|\nabla w(y)|}{|x - y|^{N-1}} dy, \quad (2.4)$$

*with  $\Omega$  a bounded domain and  $\hat{C}$  a positive constant. Let  $\rho$  be a weight function such that*

$$\int_{\Omega} \frac{1}{\rho^{\tau}|x - y|^{\gamma}} dy \leq C^* \quad \text{for any } x \in \Omega, \quad (2.5)$$

*with  $\max\{(p - 2), 0\} \leq \tau < p - 1$ ,  $\gamma < N - 2$  ( $\gamma = 0$  if  $N = 2$ ). Then*

$$\int_{\Omega} w^2 \leq C_p \int_{\Omega} \rho |\nabla w|^2, \quad (2.6)$$

*where  $C_p = C_p(d, C^*)$ , with  $d = \text{diam}(\Omega)$ . Moreover,  $C_p \rightarrow 0$  if  $d \rightarrow 0$ .*

We remark that, for the sake of simplicity and for the reader's convenience, we make explicit the dependence of  $C_p$  on the parameters, which will play a crucial role, in the sequel and that we need to control. The other parameters involved are fixed in our application and we refer the reader to Theorem 8 and to [22, Corollary 2 in Section 5] (see also [13]).

We will use the weighted Poincaré-type inequality with  $\rho = |\nabla u|^{p-2}$ . The next proposition gives some sufficient conditions for (2.5).

**Proposition 2.4.** *Let  $1 < p < \infty$  and  $u \in C^{1,\alpha}(\Omega)$  be a weak solution to*

$$-\Delta_p u = h(x) \quad \text{in } \Omega,$$

*with  $h \in W^{1,\infty}(\Omega)$ . Let  $\Omega' \subset\subset \Omega$  and  $0 < \delta < \text{dist}(\Omega', \partial\Omega)$  and assume that  $h > 0$  in  $\overline{\Omega'_\delta}$ , where*

$$\Omega'_\delta = \{x \in \Omega : d(x, \Omega') < \delta\} \Subset \Omega.$$

*Let us fix  $\beta_1, \beta_2$  such that*

$$\inf_{x \in \Omega'_\delta} h(x) \geq \beta_1 > 0 \quad \text{and} \quad \delta \geq \beta_2 > 0.$$

*Then there exists a positive constant  $C^* = C^*(\beta_1, \beta_2)$  such that*

$$\int_{\Omega'} \frac{1}{|\nabla u|^\tau} \frac{1}{|x - y|^\gamma} \leq C^*,$$

*with  $\max\{(p - 2), 0\} \leq \tau < p - 1$ .*

**Remark 2.5.** The proof of Proposition 2.4 follows from [13] (see also [36, 37]), however we refer to [22, Proposition 1 in Section 4] for the version stated here. Let us also point out that, as above, we prefer to omit the dependence of the constant  $C^*$  on other parameters that are fixed and therefore not relevant in our application.

Later we will frequently exploit the classical Harnack inequality for  $p$ -Laplace equations. We refer to [34, Theorem 7.2.1] and the references therein. At some point, as it will be clear later, it will be crucial the use of a boundary-type Harnack inequality. We thus state here an adapted version of the more general and deep result of M.F. Bidaut-Véron, R. Borghol and L. Véron, see [6, Theorem 2.8].

**Theorem 2.6 (Boundary Harnack inequality).** *Let  $R_0 > 0$  define the cylinder  $\mathcal{C}_{(0,L)}(2R_0)$  as in (2.2) and let  $u$  be such that*

$$-\Delta_p u = c(x)u^{p-1} \quad \text{in } \mathcal{C}_{(0,L)}(2R_0),$$

*with  $u$  vanishing on  $\mathcal{C}_{(0,L)}(2R_0) \cap \{y = 0\}$  and with  $\|c(x)\|_{L^\infty(\mathcal{C}_{(0,L)}(2R_0))} \leq C_0$ . Then*

$$\frac{1}{C} \frac{u(z_2)}{\rho(z_2)} \leq \frac{u(z_1)}{\rho(z_1)} \leq C \frac{u(z_2)}{\rho(z_2)}, \quad \forall z_1, z_2 \in B_{R_0} \cap \mathcal{C}_{(0,L)}(2R_0) : 0 < \frac{|z_2|}{2} \leq |z_1| \leq 2|z_2|,$$

*where  $C = C(p, N, C_0)$  and  $\rho(\cdot)$  is the distance function to  $\partial\mathbb{R}_+^N$ .*



Finally, we state a lemma that will be useful in the proof of Proposition 3.3 below, see [21, Lemma 2.1].

**Lemma 2.7.** *Let  $\theta > 0$  and  $\nu > 0$  such that  $\theta < 2^{-\nu}$ . Let*

$$\mathcal{L} : (1, +\infty) \rightarrow \mathbb{R}$$

*be a non-negative and non-decreasing function such that*

$$\begin{cases} \mathcal{L}(R) \leq \theta \mathcal{L}(2R) & \forall R > 1 \\ \mathcal{L}(R) \leq C R^\nu & \forall R > 1. \end{cases}$$

*Then  $\mathcal{L}(R) = 0$ .*

### 3. Proof of Theorem 1.1

We will give the proof of Theorem 1.1 at the end of this section. Let us begin by showing that any positive solution to (1.1) is increasing in the  $y$ -direction near the boundary  $\partial \mathbb{R}_+^N$ . We prove such a result for problems involving a more general class of nonlinearities and for any  $1 < p < \infty$ . We have the following:

**Theorem 3.1.** *Let  $1 < p < \infty$  and let  $u \in C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{R}_+^N})$  be a positive weak solution to (1.1) with  $|\nabla u| \in L^\infty(\mathbb{R}_+^N)$ . Assume that the nonlinearity  $f$  is continuous in  $\mathbb{R}^+ \cup \{0\}$  and, for some  $T > 0$ , it holds that*

$$|f(t)| \leq \bar{k} t^{p-1} \quad \text{for } t \in [0, T]$$

*for some  $\bar{k} = \bar{k}(T) > 0$ . Then it follows that there exists  $\lambda > 0$  such that*

$$\frac{\partial u}{\partial y}(x', y) > 0 \quad \text{in } \Sigma_\lambda. \quad (3.1)$$

*In particular the result holds true under the condition  $(h_f)$ .*

*Proof.* We argue by contradiction and we assume that there exists a sequence of points  $P_n = (x'_n, y_n)$  such that

$$\frac{\partial u}{\partial y}(x'_n, y_n) \leq 0 \quad \text{and} \quad y_n \xrightarrow{n \rightarrow +\infty} 0. \quad (3.2)$$

We consider the sequence  $\hat{x}_n$  defined by  $\hat{x}_n = (x'_n, 1)$ . We set  $\alpha_n = u(x'_n, 1)$ , and

$$w_n(x', y) = \frac{u(x' + x'_n, y)}{\alpha_n}. \quad (3.3)$$

We remark that  $w_n(0, 1) = 1$  and we have

$$\begin{aligned} -\Delta_p w_n(x) &= \frac{1}{\alpha_n^{p-1}} f(u(x' + x'_n, y)) \\ &= \frac{1}{\alpha_n^{p-1}} \frac{f(u(x' + x'_n, y))}{u^{p-1}(x' + x'_n, y)} u^{p-1}(x' + x'_n, y) = c_n(x) w_n^{p-1}(x), \end{aligned} \quad (3.4)$$

for

$$c_n(x) = \frac{f(u(x' + x'_n, y))}{u^{p-1}(x' + x'_n, y)}. \quad (3.5)$$

Since for any  $L > 0$  we have that  $u \in L^\infty(\Sigma_{(L)})$  (by the Dirichlet condition and because  $|\nabla u|$  is bounded in  $\mathbb{R}_+^N$ ), by the assumption on the nonlinearity  $f$ , we obtain that

$$\|c_n(x)\|_{L^\infty(\Sigma_L)} \leq \|c_n(x)\|_{L^\infty(\Sigma_{2L})} \leq C_0(L). \quad (3.6)$$

Now we consider real numbers  $L, R$  and  $R_0$  satisfying

$$0 < 2R_0 < 1 < R < L. \quad (3.7)$$

We claim that:

$$\|w_n\|_{L^\infty(\mathcal{C}_{(0,L)}(R))} \leq C(L, R, R_0).$$

Since  $w_n(0, 1) = 1$ , by the classical Harnack inequality, see [34, Theorem 7.2.1], we have that

$$\|w_n\|_{L^\infty(\mathcal{C}_{(0,L)}(R) \cap \{y \geq \frac{R_0}{4}\})} \leq C_H^i(L, R, R_0). \quad (3.8)$$

Now we apply Theorem 2.6 to deduce that

$$\|w_n\|_{L^\infty(\mathcal{C}_{(0,L)}(R) \cap \{y \leq \frac{R_0}{4}\})} \leq C_H^b(L, R, R_0). \quad (3.9)$$

To this end, let  $\tilde{P} = (\tilde{x}', \tilde{y})$  be such that  $\tilde{x}' \in B'_R(0)$  and  $0 < \tilde{y} < \frac{R_0}{4}$  and consider a corresponding point  $\check{Q} = (\check{x}', 0)$  such that

$$\check{x}' \in B'(0, R) \quad \text{and} \quad \tilde{P} \in \partial B_{R_0}(\check{Q}).$$

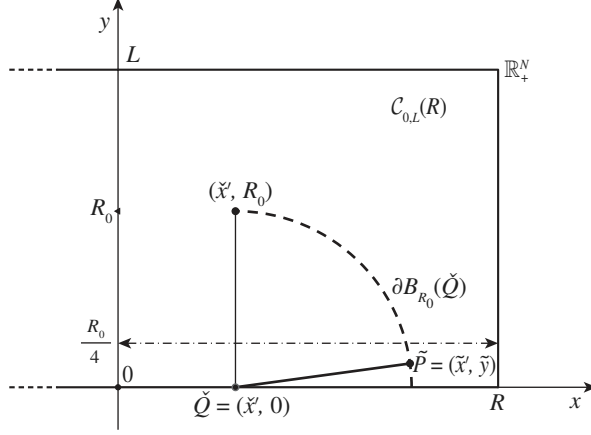
Recalling the choice  $2R_0 < R < L$ , it is easy to check that such a point exists (and in general is not unique), see Figure 3.1.

By [6] (see Theorem 2.6) and recalling (3.6), we infer that

$$\frac{w_n(\tilde{P})}{\tilde{y}} \leq C \frac{w_n(\check{x}', R_0)}{R_0},$$

and, recalling also that  $w_n(x, 0) = 0$ , we deduce that

$$\|w_n\|_{L^\infty(\mathcal{C}_{(0,L)}(R) \cap \{y \leq \frac{R_0}{4}\})} \leq \frac{C}{4} \cdot C_H^i(L, R, R_0),$$



**Figure 3.1.**

that is (3.9) holds, with  $C_H^b(L, R, R_0) = C \cdot C_H^i(L, R, R_0)$ . Finally using (3.8) and (3.9) it follows that

$$\|w_n\|_{L^\infty(\mathcal{C}_{(0,L)}(R))} \leq C(L, R, R_0).$$

Now consider  $u$  (and consequently  $u(x' + x'_n, y)$  in (3.3)), defined on the entire space  $\mathbb{R}^N$  by odd reflection. That is

$$u(x', y) = -u(x', -y) \quad \text{in } \{y < 0\},$$

and consequently

$$f(t) = -f(-t) \quad \text{if } t < 0.$$

In this case we will refer to the cylinder

$$\mathcal{C}_{(-L,L)}(R) = B'_R(0) \times (-L, L).$$

By standard regularity theory, see, e.g., [41, Theorem 1], since  $\|w_n\|_{L^\infty(\mathcal{C}_{(-L,L)}(R))} \leq C(L, R, R_0)$ , we have that

$$\|w_n\|_{C_{\text{loc}}^{1,\alpha}(\mathcal{C}_{(-L,L)}(R))} \leq C(L, R, R_0)$$

for some  $0 < \alpha < 1$ . This allows us to use the Ascoli-Arzelà theorem to get

$$w_n \xrightarrow{C_{\text{loc}}^{1,\alpha'}(\mathcal{C}_{(-L,L)}(R))} w_0$$

up to subsequences, for  $\alpha' < \alpha$ . Furthermore, thanks to (3.6), we infer that

$$c_n(\cdot) \rightarrow c_0(\cdot) \tag{3.10}$$

weakly star in  $L^\infty(\mathcal{C}_{(-L,L)}(R))$  up to subsequences. This and the fact that  $w_0 \in C^{1,\alpha'}(\mathcal{C}_{(-L,L)}(R))$  allows us to deduce easily that

$$\begin{cases} -\Delta_p w_0 = c_0(x) w_0^{p-1} & \text{in } \mathcal{C}_{(0,L)}(R) \\ w_0(x', y) \geq 0 & \text{in } \mathcal{C}_{(0,L)}(R) \\ w_0(x', 0) = 0 & \text{on } \partial \mathcal{C}_{(0,L)}(R) \cap \partial \mathbb{R}_+^N. \end{cases}$$

By the strong maximum principle, and recalling that  $w_n(0, 1) = 1$  for all  $n \in \mathbb{N}$ , we deduce that  $w_0 > 0$  in  $\mathcal{C}_{(0,L)}(R)$  and, by Hopf's lemma, we infer that

$$\frac{\partial w_0}{\partial y}(0, 0) > 0.$$

We conclude the proof noticing that a contradiction occurs since by (3.2) we should have that  $\frac{\partial w_0}{\partial y}(0, 0) \leq 0$ .  $\square$

**Corollary 3.2.** *Under the hypotheses of Theorem 3.1, there exists  $\lambda > 0$  such that, for all  $0 < \theta \leq \frac{\lambda}{2}$ , it holds that*

$$u \leq u_\theta \quad \text{in } \Sigma_\theta.$$

*Proof.* Given  $\lambda$  from Theorem 3.1, using (3.1), it is sufficient to recall the definition of  $u_\theta$  in (2.3).  $\square$

We now prove a technical result we are going to use in the sequel to prove our main result; we may refer to it as a weak comparison principle in narrow domains. We define the projection  $\mathcal{P}$  as

$$\begin{aligned} \mathcal{P} : \mathbb{R}^N &\longrightarrow \mathbb{R}^{N-1} \\ (x', y) &\longrightarrow x'. \end{aligned}$$

In the proof of the next proposition, we will use the following inequalities:  $\forall \eta, \eta' \in \mathbb{R}^N$  with  $|\eta| + |\eta'| > 0$  there exists positive constants  $\dot{C}, \check{C}$  depending on  $p$  such that

$$\begin{aligned} [|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'] [\eta - \eta'] &\geq \dot{C} (|\eta| + |\eta'|)^{p-2} |\eta - \eta'|^2, \\ | |\eta|^{p-2}\eta - |\eta'|^{p-2}\eta' | &\leq \check{C} (|\eta| + |\eta'|)^{p-2} |\eta - \eta'|. \end{aligned} \tag{3.11}$$

**Proposition 3.3.** *Let  $p > 2$  and let  $u \in C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{R}_+^N})$  be a positive weak solution to (1.1) with  $|\nabla u| \in L^\infty(\mathbb{R}_+^N)$ . For  $0 \leq \alpha < \beta \leq \lambda$ , let  $\Sigma_{(\alpha,\beta)}$  be the strip defined in (2.1) and assume that*

$$u \leq u_\lambda \quad \text{on } \partial \Sigma_{(\alpha,\beta)}. \tag{3.12}$$

Assume furthermore that, setting

$$\mathcal{I}_{(\lambda)}^+ = \left\{ (x', \lambda) : x' \in \mathcal{P}(\text{supp}(u - u_\lambda)^+) \right\},$$

it holds that

$$u(x) \geq \gamma > 0 \quad \text{on} \quad \mathcal{I}_{(\lambda)}^+. \quad (3.13)$$

Then, for fixed  $\Lambda > 0$  such that  $\Lambda \geq 2\lambda + 1$ , there exists  $h_0 = h_0(f, p, \gamma, N, \|\nabla u\|_{L^\infty(\Sigma_\Lambda)})$  such that if  $\beta - \alpha \leq h_0$  we have  $u \leq u_\lambda$  in  $\Sigma_{(\alpha, \beta)}$ .

*Proof.* Recalling that  $u_\lambda(x', y) = u(x', 2\lambda - y)$ , we remark that  $(u - u_\lambda)^+ \in L^\infty(\Sigma_{(\alpha, \beta)})$  since we assumed  $|\nabla u|$  is bounded. Let us now define

$$\Psi = (u - u_\lambda)^+ \varphi_R^2,$$

where  $\varphi_R(x', y) = \varphi_R(x') \in C_c^\infty(\mathbb{R}^{N-1})$ ,  $\varphi_R \geq 0$  such that

$$\begin{cases} \varphi_R \equiv 1 & \text{in } B'(0, R) \subset \mathbb{R}^{N-1} \\ \varphi_R \equiv 0 & \text{in } \mathbb{R}^{N-1} \setminus B'(0, 2R) \\ |\nabla \varphi_R| \leq \frac{C}{R} & \text{in } B'(0, 2R) \setminus B'(0, R) \subset \mathbb{R}^{N-1}, \end{cases} \quad (3.14)$$

where  $B'(0, R)$  denotes the ball in  $\mathbb{R}^{N-1}$  with center 0 and radius  $R > 0$ . From now on, for the sake of simplicity, we set  $\varphi_R(x', y) := \varphi(x', y)$ . By (3.14) and since  $u \leq u_\lambda$  on  $\partial \Sigma_{(\lambda, \beta)}$  (see (3.12)), it follows that  $\Psi \in W_0^{1,p}(\mathcal{C}_{(\alpha, \beta)}(2R))$ . Since  $u$  is a solution to problem (1.1), then it follows that  $u, u_\lambda$  are solutions to

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Sigma_{(\alpha, \beta)} \\ -\Delta_p u_\lambda = f(u_\lambda) & \text{in } \Sigma_{(\alpha, \beta)} \\ u \leq u_\lambda & \text{on } \partial \Sigma_{(\alpha, \beta)}. \end{cases} \quad (3.15)$$

Then using  $\Psi$  as a test function in both equations of problem (3.15) and subtracting we get

$$\begin{aligned} & \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla (u - u_\lambda)^+) \varphi^2 \\ & + \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla \varphi^2) (u - u_\lambda)^+ \\ & = \int_{\mathcal{C}(2R)} (f(u) - f(u_\lambda)) (u - u_\lambda)^+ \varphi^2, \end{aligned} \quad (3.16)$$

where  $\mathcal{C}(\cdot)$  denotes the cylinder defined in (2.2). By (3.11) and the fact that  $p \geq 2$ , from (3.16) we deduce that

$$\begin{aligned}
& \dot{\mathcal{C}} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2 \varphi^2 \\
& \leq \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla(u - u_\lambda)^+) \varphi^2 \\
& = - \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla \varphi^2) (u - u_\lambda)^+ \\
& \quad + \int_{\mathcal{C}(2R)} (f(u) - f(u_\lambda)) (u - u_\lambda)^+ \varphi^2 \\
& \leq \int_{\mathcal{C}(2R)} \left| (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla \varphi^2) \right| (u - u_\lambda)^+ \\
& \quad + \int_{\mathcal{C}(2R)} (f(u) - f(u_\lambda)) (u - u_\lambda)^+ \varphi^2 \\
& \leq \check{C} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+| |\nabla \varphi^2| (u - u_\lambda)^+ \\
& \quad + \int_{\mathcal{C}(2R)} (f(u) - f(u_\lambda)) (u - u_\lambda)^+ \varphi^2,
\end{aligned} \tag{3.17}$$

where in the last line we used the Schwarz inequality and the second of (3.11). Setting

$$I_1 := \check{C} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+| |\nabla \varphi^2| (u - u_\lambda)^+ \tag{3.18}$$

and

$$I_2 := \int_{\mathcal{C}(2R)} (f(u) - f(u_\lambda)) (u - u_\lambda)^+ \varphi^2, \tag{3.19}$$

equation (3.17) becomes

$$\dot{\mathcal{C}} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2 \varphi^2 \leq I_1 + I_2. \tag{3.20}$$

In order to estimate the terms  $I_1$  and  $I_2$  in (3.20) we will use the weighted Poincaré-type inequality (2.6) (see [13]) and a covering argument that goes back to [22]. Let us consider the hypercubes  $Q_i$  of  $\mathbb{R}^N$  defined by

$$Q_i = Q'_i \times [\alpha, \beta],$$

where  $Q'_i \subset \mathbb{R}^{N-1}$  are hypercubes of  $\mathbb{R}^{N-1}$  with edge  $\beta - \alpha$  and such that

$$\bigcup_i Q'_i = \mathbb{R}^{N-1}.$$

Moreover we assume that  $Q_i \cap Q_j = \emptyset$  for  $i \neq j$  and

$$\bigcup_{i=1}^{\overline{N}} \overline{Q_i} \supset \mathcal{C}(2R). \quad (3.21)$$

It follows that each set  $Q_i$  has diameter

$$\text{diam}(Q_i) = d_Q = \sqrt{N}(\beta - \alpha), \quad i = 1, \dots, \overline{N}. \quad (3.22)$$

The covering in (3.21) will allow us to use in each  $Q_i$  the weighted Poincaré-type inequality and to take advantage of the constant  $C_p$  in Theorem 2.3, that turns out not to be depending on the index  $i$  of (3.21). We will recollect the estimates later.

Let us define

$$w(x) = \begin{cases} (u - u_\lambda)^+(x', y) & \text{if } (x', y) \in \overline{Q_i} \\ -(u - u_\lambda)^+(x', 2\beta - y) & \text{if } (x', y) \in \overline{Q_i^r}, \end{cases} \quad (3.23)$$

where  $(x', y) \in \overline{Q_i^r}$  if and only if  $(x', 2\beta - y) \in \overline{Q_i}$ . We claim that

$$\int_{Q_i} w^2 \leq C_p(Q_i) \int_{Q_i} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w|^2 \quad (3.24)$$

where  $C_p(Q_i)$  is given by Theorem 2.3 and it goes to zero if the diameter of  $Q_i$  does. Actually, since  $p \geq 2$ , we will deduce (3.24) from

$$\int_{Q_i} w^2 \leq C_p(Q_i) \int_{Q_i} |\nabla u_\lambda|^{p-2} |\nabla w|^2. \quad (3.25)$$

The fact that Theorem 2.3 can be applied to deduce (3.25) is somewhat technical and we describe the procedure below.

We have  $\int_{Q_i \cup Q_i^r} w(x) dx = 0$  and therefore, see [29, Lemma 7.14, Lemma 7.16], it follows that

$$w(x) = \hat{C} \int_{Q_i \cup Q_i^r} \frac{(x_i - z_i) D_i w(z)}{|x - z|^N} dz \quad \text{a.e. } x \in Q_i \cup Q_i^r,$$

where  $\hat{C} = \hat{C}(d_Q, N)$ , is a positive constant. Then for almost every  $x \in Q_i$  we have

$$\begin{aligned} |w(x)| &\leq \hat{C} \int_{Q_i \cup Q_i^r} \frac{|\nabla w(z)|}{|x - z|^{N-1}} dz \\ &= \hat{C} \int_{Q_i} \frac{|\nabla w(z)|}{|x - z|^{N-1}} dz + \hat{C} \int_{Q_i^r} \frac{|\nabla w(z)|}{|x - z|^{N-1}} dz \\ &\leq 2\hat{C} \int_{Q_i} \frac{|\nabla w(z)|}{|x - z|^{N-1}} dz, \end{aligned}$$

where in the last inequality we used the following standard changing of variables

$$(z^t)' = z' \quad \text{and} \quad z_N^t = 2\beta - z_N,$$

the fact that for  $x \in Q_i$  it holds that  $(|x - z|)|_{z \in Q_i} \leq (|x - z^t|)|_{z \in Q_i}$ , and that by (3.23) it holds that  $|\nabla w(z)| = |\nabla w(z^t)|$ .

Hence (2.4) holds and, in order to prove (3.25), we need to show that (2.5) holds with

$$\rho = |\nabla u_\lambda|^{p-2}.$$

Note now that if  $w$  vanishes identically in  $Q_i$ , then there is nothing to prove. Otherwise it is easy to see that from our assumptions (see (3.13)) and the classical Harnack inequality, it follows that there exists  $\bar{\gamma} > 0$  such that

$$u \geq \bar{\gamma} > 0 \quad \text{in} \quad \tilde{Q}'_i \times [\lambda/2, 4\lambda], \quad (3.26)$$

where

$$\tilde{Q}'_i := \{x \in \mathbb{R}^{N-1} : \text{dist}(x, Q'_i) < 1\}.$$

Let us consider  $Q_i^{\mathcal{R}_\lambda}$  obtained by the reflection of  $Q_i$  with respect to the hyperplane  $T_\lambda = \{(x', y) \in \mathbb{R}^N : y = \lambda\}$ . Since  $Q_i^{\mathcal{R}_\lambda}$  is bounded away from the boundary  $\mathbb{R}^N$ , namely

$$\text{dist}(Q_i^{\mathcal{R}_\lambda}, \{y = 0\}) \geq \lambda > 0$$

because of (3.26), then Proposition 2.4 applies with

$$\beta_1 = \min_{t \in [\bar{\gamma}, \|u\|_{L^\infty(\Sigma_\Lambda)}]} f(t) \quad \text{and} \quad \beta_2 = \lambda,$$

and we obtain that

$$\int_{Q_i^{\mathcal{R}_\lambda}} \frac{1}{|\nabla u|^{p-2}} \frac{1}{|x - y|^\gamma} dy \leq C_1^*(\beta_1, \beta_2) \quad \text{for any } x \in Q_i^{\mathcal{R}_\lambda}.$$

By symmetry we deduce that

$$\int_{Q_i} \frac{1}{|\nabla u_\lambda|^{p-2}} \frac{1}{|x - y|^\gamma} dy \leq C_1^*(\beta_1, \beta_2) \quad \text{for any } x \in Q_i,$$

so that we can exploit Theorem 2.3 to obtain (3.25) and consequently (3.24).



Let us now estimate the right-hand side of (3.20). Recalling (3.18) we get

$$\begin{aligned}
I_1 &= 2\check{C} \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+| \varphi |\nabla \varphi| (u - u_\lambda)^+ \\
&= 2\check{C} \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{\frac{p-2}{2}} |\nabla(u - u_\lambda)^+| \varphi (|\nabla u| + |\nabla u_\lambda|)^{\frac{p-2}{2}} |\nabla \varphi| (u - u_\lambda)^+ \\
&\leq \delta' \check{C} \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2 \varphi^2 \\
&\quad + \frac{\check{C}}{\delta'} \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \varphi|^2 [(u - u_\lambda)^+]^2,
\end{aligned}$$

where in the last inequality we used the weighted Young inequality, with  $\delta'$  to be chosen later. Hence

$$I_1 \leq I_1^a + I_1^b, \quad (3.27)$$

where

$$\begin{aligned}
I_1^a &= \delta' \check{C} \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2 \varphi^2, \\
I_1^b &= \frac{\check{C}}{\delta'} \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \varphi|^2 [(u - u_\lambda)^+]^2.
\end{aligned} \quad (3.28)$$

Using the covering in (3.21), the properties of the cut-off function in (3.14) and the fact that  $|\nabla u|$  and  $|\nabla u_\lambda|$  are bounded, by (3.24) we deduce that

$$\begin{aligned}
I_1^b &\leq \sum_{i=1}^{\bar{N}} \frac{C}{\delta' R^2} \int_{C(2R) \cap Q_i} [(u - u_\lambda)^+]^2 \\
&\leq \max_i C_P(Q_i) \sum_{i=1}^{\bar{N}} \frac{C}{\delta' R^2} \int_{C(2R) \cap Q_i} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2 \quad (3.29) \\
&\leq C_P^* \frac{C}{\delta' R^2} \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2
\end{aligned}$$

where  $C_P^* = \max_i C_P(Q_i)$  and  $C = C(p, \|\nabla u\|_{L^\infty(\Sigma_\Lambda)})$ .

Now we estimate the term  $I_2$  in (3.20). Since  $f$  is locally Lipschitz continuous because of (3.19), arguing as in (3.29), we get that

$$\begin{aligned}
I_2 &\leq \int_{C(2R)} \frac{f(u) - f(u_\lambda)}{u - u_\lambda} [(u - u_\lambda)^+]^2 \\
&\leq C_P^* \cdot C \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2,
\end{aligned}$$

where  $C_p^*$  is as in (3.29) and  $C = C(f, \lambda, \|\nabla u\|_{L^\infty(\Sigma_\Lambda)})$ . The constant  $C$  will depend on the Lipschitz constant of  $f$  in the interval  $[0, \max\{\|u\|_{L^\infty(\Sigma_\Lambda)}, \|u_\lambda\|_{L^\infty(\Sigma_\Lambda)}\}]$ . By (3.20), (3.27), (3.28) and (3.29), up to redefining the constants, we obtain

$$\begin{aligned}
& C \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2 \varphi^2 \\
& \leq \delta' \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - v)^+|^2 \\
& \quad + \frac{C_p^*}{R} \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2 \\
& \quad + C_p^* \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2.
\end{aligned} \tag{3.30}$$

Let us choose  $\delta'$  small in (3.30) such that  $C - \delta' > C/2$  and fix  $R > 1$ . Then we obtain

$$\begin{aligned}
& \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2 \varphi^2 \\
& \leq 4 \frac{C_p^*}{C} \int_{C(2R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2.
\end{aligned} \tag{3.31}$$

To conclude we set now

$$\mathcal{L}(R) := \int_{C(R)} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla(u - u_\lambda)^+|^2. \tag{3.32}$$

We can fix  $h_0 = h_0(f, p, \gamma, \lambda, N, \|\nabla u\|_{L^\infty(\Sigma_\Lambda)})$  positive, such that if

$$\beta - \alpha \leq h_0,$$

(recall that  $C_p^* \rightarrow 0$  in this case since  $\text{diam}(Q_i) \rightarrow 0$ , see (3.22)) then

$$\theta := 4 \frac{C_p^*}{C} < 2^{-N}.$$

Then, by (3.31) and (3.32), we have

$$\begin{cases} \mathcal{L}(R) \leq \theta \mathcal{L}(2R) & \forall R > 1 \\ \mathcal{L}(R) \leq C R^N & \forall R > 1. \end{cases}$$

From Lemma 2.7 with  $v = N$  and  $\theta < 2^{-N}$ , we get

$$\mathcal{L}(R) \equiv 0$$

and consequently that  $(u - u_\lambda)^+ \equiv 0$ . □

The proof of our main result will follow by the moving-plane procedure, strongly based on Proposition 3.3. As it will be clear later, it will be needed to substitute  $\lambda$  by  $\lambda + \varepsilon$  in order to proceed further from the maximal position. To do this we need to be very accurate in the estimate of the constants involved, namely we need to control the role of  $h_0$  in Proposition 3.3. This is the reason why we introduced the larger strip  $\Sigma_\Lambda$ , that allows us to control  $|\nabla u|$ . But we still need to control the dependence of  $h_0$  on  $\gamma$  (see (3.13)). Equivalently we need a uniform control (with respect to  $\varepsilon$ ) on the infimum of  $u$  far from the boundary, and in the set where  $u$  is greater than  $u_\lambda$ . This motivates the following:

**Lemma 3.4.** *Let  $\lambda > 0$  and let  $u$  be a solution to (1.1), with  $|\nabla u| \in L^\infty(\mathbb{R}_+^N)$  and  $u_\lambda$  defined as in (2.3). Assume here that  $(h_f)$  is fulfilled with  $f_0 = 0$  and define*

$$\mathcal{I}_{(\lambda, \varepsilon)}^+ = \left\{ (x', \lambda) : x' \in \mathcal{P}(\text{supp}(u - u_{\lambda+\varepsilon})^+) \right\}.$$

*Then there exist  $\varepsilon_0 > 0$  and  $\gamma > 0$  such that*

$$u(x) \geq \gamma \quad \text{on} \quad \mathcal{I}_{(\lambda, \varepsilon)}^+,$$

*for all  $0 \leq \varepsilon \leq \varepsilon_0$ .*

*Proof.* By contradiction, given  $\varepsilon_0 > 0$  and  $\gamma > 0$ , we find  $0 \leq \varepsilon \leq \varepsilon_0$  and a point  $Q_\varepsilon = (x'_\varepsilon, \lambda)$  with  $Q_\varepsilon \in \mathcal{I}_{(\lambda, \varepsilon)}^+$  such that

$$u(x'_\varepsilon, \lambda) \leq \gamma.$$

It is convenient to consider  $\varepsilon_0 = \gamma = 1/n$  and the corresponding  $\varepsilon = \varepsilon_n \leq \varepsilon_0$  defined by contradiction as above, that obviously approaches zero as  $n$  tends to infinity. Also we use the notation  $Q_{\varepsilon_n} \in \mathcal{I}_{(\lambda, \varepsilon_n)}^+$ . On a corresponding sequence  $P_n = (x'_n, y_n)$  we have that

$$u(x'_n, y_n) \geq u_{\lambda+\varepsilon_n}(x'_n, y_n) \quad \text{with} \quad (x'_n, y_n) \in \Sigma_{\lambda+\varepsilon_n}, \quad (3.33)$$

where the existence of the sequence  $(x'_n, y_n)$  follows by the fact that  $Q_{\varepsilon_n} \in \mathcal{I}_{(\lambda, \varepsilon_n)}^+$  and (up to subsequences)  $y_n \rightarrow y_0 \in [0, \lambda]$ . Moreover  $\lim_{n \rightarrow +\infty} u(x'_n, \lambda) \rightarrow 0$ . Let us set

$$w_n(x', y) = \frac{u(x' + x'_n, y)}{\alpha_n} \quad (3.34)$$

and  $\alpha_n = u(x'_n, \lambda)$ , with  $\lim_{n \rightarrow +\infty} \alpha_n = 0$ . We remark that  $w_n(0, \lambda) = 1$ . Then we have

$$-\Delta_p w_n(x) = c_n(x) w_n^{p-1}(x), \quad (3.35)$$

for

$$c_n(x) = \frac{f(u(x' + x'_n, y))}{u^{p-1}(x' + x'_n, y)}. \quad (3.36)$$

Since for any  $L > 0$  we have that  $u \in L^\infty(\Sigma_{(L)})$  (by the Dirichlet condition and because  $|\nabla u|$  is bounded in  $\mathbb{R}_+^N$ ), by  $(h_f)$  we obtain that

$$\|c_n(x)\|_{L^\infty(\Sigma_L)} \leq C(L). \quad (3.37)$$

For  $L > \lambda$  we consider the cylinder  $\mathcal{C}_{(0,L)}(R)$  and, arguing as in the proof of Theorem 3.1 (see the first claim there), we deduce that

$$\|w_n\|_{L^\infty(\mathcal{C}_{(0,L)}(R))} \leq C(L).$$

Now, as in the proof of Theorem 3.1, we consider  $u$  defined on the entire space  $\mathbb{R}^N$  by odd reflection and, by standard regularity theory (see [18,41]), we deduce that

$$\|w_n\|_{C_{\text{loc}}^{1,\alpha}(\mathcal{C}_{(-L,L)}(R))} \leq C(L)$$

for some  $0 < \alpha < 1$ . This allows us to use the Ascoli-Arzelà theorem and get

$$w_n \xrightarrow{C_{\text{loc}}^{1,\alpha'}(\mathcal{C}_{(-L,L)}(R))} w_{L,R}$$

up to subsequences, for  $\alpha' < \alpha$ . Replacing  $L$  by  $L + n$  ( $n \in \mathbb{N}$ ), and  $R$  by  $R + n$ , we can repeat the argument above and then perform a standard diagonal process to define  $w$  in the entire space  $\mathbb{R}^N$  in such a way that  $w$  is locally the limit of subsequences of  $w_n$ . It turns out that, by construction, setting

$$w_+(x) = w(x) \cdot \chi_{\overline{\mathbb{R}_+^N}}$$

we have that

$$\begin{cases} -\Delta_p w_+ = 0 & \text{in } \mathbb{R}_+^N \\ w_+(x', y) \geq 0 & \text{in } \mathbb{R}_+^N \\ w_+(x', 0) = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases}$$

This is a simple computation where in (3.35) we need to use the fact that  $c_n(x) \rightarrow 0$  as  $n \rightarrow +\infty$  uniformly on compact sets. This follows considering that  $w_n$  is uniformly bounded on compact sets and then, by (3.34), it follows that  $u(x + x'_n, y) \rightarrow 0$  as  $n \rightarrow +\infty$ . By (3.36) and recalling that

$$\lim_{t \rightarrow 0} \frac{f(t)}{t^{p-1}} = 0,$$

finally it follows that  $c_n(x) \rightarrow 0$  on compact sets.

By the strong maximum principle we have now that  $w_+ > 0$ , in view of the fact that (by uniform convergence of  $w_n$ )  $w_+(0, \lambda) = 1$ . By [30, Theorem 3.1], it follows that  $w_+$  must be affine linear, *i.e.*,  $w_+(x', y) = ky$ , for some  $k > 0$  by the

Dirichlet condition. If  $y_0 \in [0, \lambda)$ , by (3.33) and by the uniform convergence of  $w_n \rightarrow w_+$ , we would have

$$w_+(0, y_0) \geq (w_+)_\lambda(0, y_0).$$

This is a contradiction since  $w_+(x', y) = ky$  for some  $k > 0$ .

Therefore let us assume that  $y_n \rightarrow \lambda$  and note that, by the mean value theorem, at some point  $\xi_n$  lying on the segment from  $(0, y_n)$  to  $(0, 2(\lambda + \varepsilon_n) - y_n)$ , it should hold that

$$\frac{\partial w_n}{\partial y}(0, \xi_n) \leq 0.$$

Since  $w_n \rightarrow w_+$  in  $C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{R}_+^N})$  we would have that

$$\frac{\partial w_+}{\partial y}(0, \lambda) \leq 0.$$

Again this is a contradiction since  $w_+(x', y) = ky$ , for some  $k > 0$ , and the result is proved.  $\square$

The results proved above allow us to conclude the proof of our main result.

*Proof of Theorem 1.1.* We consider here the case when  $(h_f)$  is fulfilled with  $f_0 = 0$  since in the simpler case  $f_0 > 0$  the result follows directly by [22, Theorem 3]. Thanks to Corollary 3.2 we have that the set

$$\Lambda \equiv \{t > 0 : u \leq u_\alpha \text{ in } \Sigma_\alpha \quad \forall \alpha \leq t\}$$

is not empty. To conclude the proof, if we set

$$\bar{\lambda} = \sup \Lambda,$$

which now is well defined, we have to show that  $\bar{\lambda} = +\infty$ . By contradiction assume that  $\bar{\lambda} < +\infty$  and set

$$W_\varepsilon^+ := (u - u_{\bar{\lambda}+\varepsilon})^+ \chi_{\Sigma_{\bar{\lambda}+\varepsilon}}.$$

We point out that given  $0 < \delta < \bar{\lambda}/2$ , there exists  $\varepsilon_0$  such that for all  $0 < \varepsilon \leq \varepsilon_0$  it follows that

$$\text{supp } W_\varepsilon^+ \subset \Sigma_\delta \cup \Sigma_{(\bar{\lambda}-\delta, \bar{\lambda}+\varepsilon)}.$$

This follows by an analysis of the limiting profile at infinity. We do not add the details since the proof is exactly the one in [21, Proposition 4.1]. For  $\delta$  and  $\varepsilon_0$  sufficiently small Proposition 3.3 applies in  $\Sigma_\delta$  and in  $\Sigma_{(\bar{\lambda}-\delta, \bar{\lambda}+\varepsilon)}$  with  $\lambda = \bar{\lambda} + \varepsilon$  and  $\Lambda = 2\bar{\lambda} + 1$ . It is crucial here the fact that, thanks to Lemma 3.4, the parameter  $h_0$  in the statement of Proposition 3.3 can be chosen independently of  $\varepsilon$  since there  $\gamma$  does not depend on  $\varepsilon$ . Then we conclude that  $W_\varepsilon^+ \equiv 0$ . This is a contradiction with

the definition of  $\bar{\lambda}$ , so we have proved that  $\bar{\lambda} = \infty$ . This implies the monotonicity of  $u$  in the half-space, that is  $\frac{\partial u}{\partial y}(x) \geq 0$  in  $\mathbb{R}_+^N$ . By Theorem 2.2, since  $u$  is not trivial, it follows

$$\frac{\partial u}{\partial y}(x) > 0 \quad \text{in } \mathbb{R}_+^N.$$

Finally, to prove that  $u \in C_{\text{loc}}^{2,\alpha'}(\overline{\mathbb{R}_+^N})$  just note that from the fact that  $\frac{\partial u}{\partial y} > 0$  we deduce that the set of critical points  $\{\nabla u = 0\}$  is empty and consequently the equation is no more degenerate. The  $C^{2,\alpha'}$  regularity follows therefore by standard regularity results, see [29].  $\square$

*Proof of Theorem 1.2.* By [21, Theorem 1.7] it follows that  $0 < u \leq t_0$ . Thanks to the behaviour of the nonlinearity near  $t_0$  (see (1.2)), the strong maximum principle applies and implies that actually  $0 < u < t_0$  in the half space. Arguing now as in the proof of [23, Theorem 1.3] it follows that  $u$  is strictly bounded away from  $t_0$  in  $\Sigma_\lambda$  for any  $\lambda > 0$ . Now the monotonicity of the solution follows by our Theorem 1.1 (in the case  $f_0 > 0$  the result follows also directly by [22, Theorem 3]). Note in fact that the condition  $(h_f)$  is satisfied in the range of values that the solutions takes in any strip and this is sufficient in order to run over again the moving plane procedure.  $\square$

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