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# Asymptotic expansion of the solutions to a regularized Boussinesq system (theory and numeric)

Ahmad SAFA <sup>\*1,3</sup>, Hervé LE MEUR <sup>†1</sup>, Jean-Paul CHEHAB <sup>‡1</sup>, and Raafat TALHOUK <sup>§2,3</sup>

<sup>1</sup>Laboratoire LAMFA (UMR CNRS 7352), Université de Picardie Jules Verne, 33 rue Saint Leu, 80039 Amiens Cédex, France.

<sup>2</sup>Léonard de Vinci Pôle Universitaire, Research Center, 92 916 Paris La Défense, France.

<sup>3</sup>Laboratoire de mathématiques-EDST, Faculté des sciences et EDST, Université Libanaise, Hadat, Liban.

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## Abstract

We here consider the propagation of surface water waves described by the Boussinesq system. Following [9], we introduce a regularized Boussinesq system obtained by adding a non-local pseudo-differential operator define by  $\widehat{g_\lambda[\zeta]} = |k|^\lambda \hat{\zeta}_k$  with  $\lambda \in ]0, 2]$ . In this paper, we display a twofold approach: First, we study theoretically the existence of an asymptotic expansion for the solution to the Cauchy problem associated to this regularized Boussinesq system with respect to the regularizing parameter  $\epsilon$ . Then, we compute the function coefficients of the expansion (in  $\epsilon$ ) numerically and verify numerically the validity of this expansion up to order 2. We also check the numerical  $L^2$  stability of the numerical algorithm.

**Keywords:** Boussinesq system, Energy estimate, Cauchy problem, numerical stability, numerical Fourier transform, numerical expansion.

**Mathematics Subject Classification numbers:** 35Q35, 35L56, 35B30, 35B40, 65T99.

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\*E-mail: ahmad.safa@u-picardie.fr

†E-mail: herve.le.meur@u-picardie.fr

‡E-mail: jean-paul.chehab@u-picardie.fr

§E-mail: raafat.talhouk@devinci.fr

# 1 Introduction

## 1.1 Motivation

Flows with a free surface are a very difficult issue, both for the time evolution of the fluid and for the time evolution of the free surface. First, one must settle the geometry and then find a reasonable mathematical model for the flow. The latter will be the Euler equation most of the time. Then one must decide the boundary condition (slipping on the solid boundary most of the time).

Solving the full Euler equation with a free boundary is too complex. To analyze and simulate simpler models one usually derive an asymptotic model from the Euler equation, by selecting of a physical regime. To this end, a rescaling is applied, upon hypothesis on the magnitude physical parameters, introducing a small parameter on which the asymptotic model will be built. For the Boussinesq system for long wave in shallow water, we refer to [5] and [6].

In [10], Schonbek proved that the classical Boussinesq system has a weak solution. Then, Amick showed, in [2], the existence and uniqueness of a smooth solution of the same Boussinesq system. For the general case, the authors studied, in [7], the abcd-Boussinesq system for different values of parameters  $a, b, c$  and  $d$ . In particular, they prove, for all  $T > 0$  and  $s > 1$ , the existence and uniqueness of the solution of the classical Boussinesq system in  $C([0, T], H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}))$  with  $\inf_{x \in \mathbb{R}} (1 + \zeta_0) > 0$ .

Regularizing effect have been taken into consideration by several authors: either for theoretical reasons or for taking into account the regularization due to viscosity, one may add a regularizing term to Boussinesq system (see [2]). This approach was used, for instance, in [9] where the existence and uniqueness of the regularized solution is proved, and the authors establish the convergence of the regularized solution toward the non-regularized one when the regularizing parameter tends to zero.

A further question would be to know whether the regularized solution can be expanded when the regularizing parameter tends to zero. That is our goal in the present article. We will check (theoretically and numerically) that the solution to a regularized Boussinesq system has an asymptotic expansion with respect to the regularizing parameter.

The article is organized as follows. We continue this section by the presentation of the model. The second section is dedicated to the notations and preliminary estimates that will be used through the whole paper. In Section 3, we adapt the method used in [1] to prove theoretically the existence of an asymptotic expansion of the solution to the regularized Boussinesq system with respect to the regularizing parameter. First, we prove the existence of the asymptotic expansion's coefficients. Then, we prove that the rest of the asymptotic expansion is small enough. Section 4 is the numerical study and it divides into three subsections. In the first, we define the numerical scheme and we verify its convergence to the expected solution. In the second, we verify numerically the result of [9] that the regularized solution tends to the non-regularized one when the regularizing parameter  $\epsilon$  tends to zero. In the third, we check the existence of an asymptotic expansion of the regularized solution with respect to  $\epsilon$  up to order 2.

## 1.2 Presentation of the model

We consider the classical Boussinesq system:

$$\begin{cases} \zeta_t + u_x + (\zeta u)_x = 0, \\ u_t - u_{txx} + \zeta_x + uu_x = 0, \end{cases} \quad (1.1)$$

where the quantity  $u = u(x, t)$  is the horizontal velocity of the liquid particle which is at position  $x \in \mathbb{R}$  at time  $t \in \mathbb{R}^+$ , and the quantity  $\zeta = \zeta(x, t)$  is the total height of the liquid.

For  $\lambda \in ]0, 2]$  and  $\epsilon \geq 0$ , we consider the following regularized Boussinesq system:

$$\begin{cases} \zeta_t + u_x + (\zeta u)_x + \epsilon g_\lambda[\zeta] = 0, \\ u_t - u_{txx} + \zeta_x + uu_x = 0, \end{cases} \quad (1.2)$$

where the operator  $g_\lambda$  is the non-local operator defined through the Fourier transform by

$$\widehat{g_\lambda[\zeta]} = |k|^\lambda \widehat{\zeta}_k, \quad \lambda \in ]0, 2].$$

**Remark 1** For  $\lambda \in ]1, 2]$ , the operator  $g_\lambda$  can be seen as a non-local regularizing operator. While for  $\lambda \in ]0, 1]$ , this operator can be considered to model some detonation phenomena (see for instance [8]).

In [9], the authors showed that for  $s > 1/2$ ,  $\epsilon \geq 0$ ,  $\lambda \in ]0, 2]$  and for the initial condition  $(\zeta_0, u_0)$  in  $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ , there exists a unique global solution  $(\zeta_\epsilon, u_\epsilon) \in C(\mathbb{R}^+, H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}))$  of the Cauchy problem associated with (1.2).

Moreover, they prove that the solution  $(\zeta_\epsilon, u_\epsilon)$  of (1.2) with initial condition  $(\zeta_0, u_0)$  in  $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ , for any  $T > 0$ , satisfies

$$(\zeta_\epsilon, u_\epsilon) \xrightarrow{\epsilon \rightarrow 0} (\zeta, u) \quad \text{in} \quad C([0, T], H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})), \quad (1.3)$$

where  $(\zeta, u)$  denotes the solution of (1.1) with initial condition  $(\zeta_0, u_0)$ .

Last the authors showed the continuity and the convergence of the flow-map with respect to the parameter  $\epsilon \in \mathbb{R}^+$ . All these results are obtained only under the non-zero-depth condition  $1 + \zeta_0 > 0$ .

## 2 Notations and basic estimation

In this section, we define the functional spaces and we give some basic estimates.

### 2.1 Notations and functions spaces

In the following,  $C$  denotes any non-negative constant which exact expression is of no importance. The notation  $x \lesssim y$  means that there exist  $C$  such that  $x \leq Cy$ . We designate by  $C(\lambda_1, \lambda_2, \dots)$  a non-negative constant depending on the parameters  $\lambda_1, \lambda_2, \dots$  and whose dependence on the  $\lambda_j$  is

always assumed to be non-decreasing.

Let  $p$  be any constant with  $1 \leq p < \infty$ , we denote  $L^p = L^p(\mathbb{R})$  the space of all Lebesgue-measurable functions  $f$  with the standard norm

$$|f|_{L^p} = \left( \int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

For  $p = 2$ , the real inner product of any functions  $f$  and  $g$  in the Hilbert space  $L^2(\mathbb{R})$  is denoted by

$$(f, g)_{L^2} = \int_{\mathbb{R}} f(x)g(x)dx.$$

The space  $L^\infty = L^\infty(\mathbb{R})$  consists of all essentially bounded, Lebesgue-measurable functions  $f$  with the norm

$$|f|_{L^\infty} = \text{ess sup } |f(x)| < \infty.$$

For any real constant  $s$ , we define the pseudo-differential operator  $\Lambda^s$  by  $\Lambda^s = (1 - \partial_{xx})^{\frac{s}{2}}$ . The space  $H^s = H^s(\mathbb{R})$  denotes the Sobolev space of all tempered distributions  $f$  with the norm

$$|f|_{H^s} = |\Lambda^s f|_{L^2} < \infty.$$

For  $(X, |\cdot|_X)$  a Banach space, we denote as usually  $L^p([0, T], X)$ ,  $1 \leq p \leq +\infty$ , the space of measurable functions  $f$  with the norm

$$|f|_{L_t^p X} = \left( \int_0^T |f(\cdot, t)|_X^p dt \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$

and for  $p = +\infty$

$$|f|_{L_t^\infty X} = \text{ess sup}_{t \in [0, T]} |f(\cdot, t)|_X.$$

The space  $C^k(\mathbb{R})$  consists of  $k$ -times continuously differentiable functions. And  $C^k([0, T], X)$  denote the space of  $k$ -times continuously differentiable functions  $f$  from  $[0, T]$  with value in  $X$ , with the norm

$$|f|_{C^k([0, T], X)} = \max_{i \in [0, k]} \sup_{t \in [0, T]} |f^{(i)}(\cdot, t)|_X.$$

Finally, for  $f, g$  and  $fg$  belonging to Banach space  $X$  of functions, and for a closed operator  $J$  defined on  $X$ , the commutator  $[J, f]$  is defined by

$$[J, f]g = J(fg) - fJ(g).$$

## 2.2 Preliminary estimates

In the following, we recall some product and commutator estimates for functions defined on  $\mathbb{R}$ .

Let  $r_1, r_2$  in  $\mathbb{R}$  be such that  $r_1 + r_2 \geq 0$ . Then for all  $r$ , such that  $r \leq r_i$  ( $i = 1, 2$ ) and  $r < r_1 + r_2 - \frac{1}{2}$ , and all  $f \in H^{r_1}(\mathbb{R})$ ,  $g \in H^{r_2}(\mathbb{R})$  one has  $fg \in H^r(\mathbb{R})$  and

$$|fg|_{H^r} \lesssim |f|_{H^{r_1}} |g|_{H^{r_2}}. \quad (2.1)$$

For any  $r \geq 0$ , and for  $(f, g) \in (H^r(\mathbb{R}) \cap L^\infty(\mathbb{R}))^2$

$$|fg|_{H^r} \lesssim |f|_{L^\infty} |g|_{H^r} + |f|_{H^r} |g|_{L^\infty}. \quad (2.2)$$

For  $f$  and  $g$  two functions in  $W^{1,p}(\mathbb{R})$ , we have (see [4]- p. 133)

$$(f_x, g)_{L^2} = -(f, g_x)_{L^2}. \quad (2.3)$$

Recall that for any  $r \geq 0$  and  $(f, g) \in H^r(\mathbb{R})^2$ , we have by definition

$$[\Lambda^r, f]g = \Lambda^r(fg) - f\Lambda^r(g). \quad (2.4)$$

Concerning commutator estimates, we need the following proposition:

**Proposition 2.1** *Let  $b > 1/2$ ,  $r \geq 0$ . If  $f \in H^r \cap H^{b+1}$  then for all  $g \in H^{r-1} \cap H^b$ , we have*

$$|[\Lambda^r, f]g|_{L^2} \lesssim |f_x|_{L^\infty} |g|_{H^{r-1}} + |f_x|_{H^{r-1}} |g|_{L^\infty}. \quad (2.5)$$

$$|\partial_x([\Lambda^r, f]g)|_{L^2} \lesssim |[\Lambda^r, f_x]g|_{L^2} + |[\Lambda^r, f]g_x|_{L^2}. \quad (2.6)$$

Moreover, if  $0 \leq r \leq b+1$ ,  $f \in H^{b+1}$  and  $g \in H^{r-1}$ , one has

$$|[\Lambda^r, f]g|_{L^2} \lesssim |f_x|_{H^b} |g|_{H^{r-1}}. \quad (2.7)$$

See [3]- pp. 283 and 286 for the proofs of (2.1), (2.2), (2.5) and (2.7). Estimate (2.6) is a consequence of the following equality:

$$\begin{aligned} \partial_x([\Lambda^r, f]g) &= \partial_x(\Lambda^r(fg)) - \partial_x(f\Lambda^r g) \\ &= \Lambda^r(f_x g) - f_x \Lambda^r g + \Lambda^r(f g_x) - f \Lambda^r g_x \\ &= [\Lambda^r, f_x]g + [\Lambda^r, f]g_x. \end{aligned}$$

□

### 3 Expansion of the regularized solution with respect to $\epsilon$

In this section, we prove the existence of an asymptotic expansion, with respect to  $\epsilon$ , of the solution  $(\zeta_\epsilon, u_\epsilon)$  to the Cauchy problem associated with the system (1.2).

Let  $\epsilon > 0$ ,  $\lambda \in ]0, 2]$  and

$$s > 1/2 + \max(\lambda, 1). \quad (3.1)$$

For  $m \in \mathbb{N}^*$  we define

$$\alpha_l^m = s + (m - l) \max(\lambda, 1), \quad 1 \leq l \leq m. \quad (3.2)$$

We now give our main Theorem:

**Theorem 3.1** For  $T > 0$  and  $(\zeta_0, u_0) \in H^{\alpha_0^m}(\mathbb{R}) \times H^{\alpha_0^m+1}(\mathbb{R})$ , let  $(\zeta_\epsilon, u_\epsilon)$ , respectively  $(\zeta, u)$ ,  $\in C([0, T], H^{\alpha_0^m}(\mathbb{R}) \times H^{\alpha_0^m+1}(\mathbb{R}))$  be the solution to the Cauchy problem associated with (1.2), respectively with (1.1), and initial data  $(\zeta_0, u_0)$ . Then there exist  $m$  vectors  $(\zeta_l, u_l)$  in  $C([0, T], H^{\alpha_l^m}(\mathbb{R}) \times H^{\alpha_l^m+1}(\mathbb{R}))$  such that,  $(\zeta_\epsilon, u_\epsilon)$ , admits in  $C([0, T], H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}))$ , an asymptotic expansion up to order  $m$ , given by

$$\zeta_\epsilon = \zeta + \sum_{l=1}^m \epsilon^l \zeta_l + o(\epsilon^m), \quad u_\epsilon = u + \sum_{l=1}^m \epsilon^l u_l + o(\epsilon^m). \quad (3.3)$$

**Remark 2** For  $f$  in a Banach space  $X$ , the error  $f = o(\epsilon^m)$  is defined by

$$\frac{\|f\|_X}{\epsilon^m} \longrightarrow 0 \quad \text{when} \quad \epsilon \longrightarrow 0.$$

To prove Theorem 3.1, we first prove the existence of  $m$  vectors  $(\zeta_l, u_l)$ ,  $1 \leq l \leq m$  in Section 3.1. Then, we prove that these  $m$  vectors are the coefficients of the asymptotic expansion (3.3) in section 3.2.

Throughout this paper we will use the following elementary result:

**Proposition 3.1** Let  $m \in \mathbb{N}^*$  and  $1 \leq l \leq m$ . For  $i = 0, \dots, l-1$ , we have

$$\alpha_i^m \geq \alpha_l^m + \max(\lambda, 1),$$

and then,  $H^{\alpha_i^m}(\mathbb{R})$  is continuously injected in  $H^{\alpha_l^m+1}(\mathbb{R}) \cap H^{\alpha_l^m+\lambda}(\mathbb{R})$ .

**Proof:**

$$\begin{aligned} \alpha_i^m &= s + (m - i) \cdot \max(\lambda, 1) \\ &\geq s + (m - l) \max(\lambda, 1) + \max(\lambda, 1), \quad \text{because} \quad i \leq l - 1 \\ &\geq \alpha_l^m + \max(\lambda, 1), \end{aligned}$$

and then, the result follows. □

### 3.1 Existence of the asymptotic expansion's coefficients

We put the assumed expansion (3.3) of  $(\zeta_\epsilon, u_\epsilon)$  in the system (1.2). The system satisfied by  $(\zeta_l, u_l)$  is

$$\text{for } 1 \leq l \leq m, \quad \begin{cases} (\zeta_l)_t + (u_l)_x + (\zeta_l \cdot u)_x + (\zeta \cdot u_l)_x = f_{1,l}, \\ (u_l)_t - (u_l)_{txx} + (\zeta_l)_x + (u \cdot u_l)_x = f_{2,l}, \\ \zeta_l(x, 0) =: \zeta_{l0}, \quad u_l(x, 0) =: u_{l0}, \end{cases} \quad (3.4)$$

where

$$f_{1,l} = - \sum_{i=1}^{l-1} (\zeta_i u_{l-i})_x - g_\lambda[\zeta_{l-1}], \quad \text{and} \quad f_{2,l} = - \sum_{i=1}^{l-1} u_i \cdot (u_{l-i})_x. \quad (3.5)$$

So, the solution  $(\zeta_l, u_l)$  are defined owing to the preliminary knowledge of previous order terms in the expansion  $(\zeta_i, u_i)_{i=1, \dots, l-1}$  and also  $(\zeta, u)$ . We will make an induction to prove all those coefficients exist and later check the difference between these coefficients is  $o(1)$ .

**Remark 3** In our problem,  $\zeta_{l0} = u_{l0} = 0$  but we want to show the existence in the general case independently of the value of  $\zeta_{l0}$  and  $u_{l0}$ .

In this subsection, we prove the existence and regularity of the solution  $(\zeta_l, u_l)$  of the Cauchy problem (3.4), assuming  $(\zeta_i, u_i)_{i=1, \dots, l-1}$  as stated in the following Theorem 3.2.

**Theorem 3.2** Let  $T > 0$ ,  $\epsilon > 0$ ,  $\lambda \in ]0, 2]$  and  $s > 1/2 + \max(\lambda, 1)$ . We suppose  $(\zeta_{l0}, u_{l0}) \in H^{\alpha_l^m}(\mathbb{R}) \times H^{\alpha_l^m+1}(\mathbb{R})$  and the functions  $(\zeta_i, u_i)_{i=1, \dots, l-1}$  are in  $C([0, T], H^{\alpha_i^m}(\mathbb{R}) \times H^{\alpha_i^m+1}(\mathbb{R}))$  with  $\alpha_i^m$ ,  $i = 1, \dots, m$  defined in (3.2).

Then there exists, a unique solution  $(\zeta_l, u_l)$  of system (3.4) in  $C([0, T], H^{\alpha_l^m}(\mathbb{R}) \times H^{\alpha_l^m+1}(\mathbb{R}))$ . This solution satisfies for all  $t \in [0, T]$ , the following inequality:

$$E^\alpha(\zeta_l, u_l) \lesssim e^{A \cdot t/2} E^\alpha(\zeta_{l0}, u_{l0}) + \frac{1}{2} \int_0^t e^{A(t-r)/2} B(r) dr. \quad (3.6)$$

where  $B(r)$  depends on the functions  $(\zeta_i, u_i)_{i=1, \dots, l-1}$ .

**Proof of Theorem 3.2:**

The existence of solution of system (3.4) in  $C([0, T], H^{\alpha_l^m}(\mathbb{R}) \times H^{\alpha_l^m+1}(\mathbb{R}))$  follows, in a classical way, by using the Friedrichs regularization technique, the Cauchy-Lipschitz theorem together with energy estimate in the vain of (3.6) written for the Friedrichs mollifiers solutions (see for instance [11] p. 414 or [14] Appendix A). The uniqueness is easily showed by estimate (3.6) written for the difference of two solutions.

**Remark 4** The existence and uniqueness result concerning system (3.4) can be also obtained by Bona-Smith regularization technic always with the Cauchy-Lipschitz theorem and energy estimate of type (3.6) (see ([9]) in case of nonlinear system).

In the following we give a detailed proof of the energy estimate (3.6).

To prove the energy estimate (3.6), we define, for  $\alpha > 0$ , the energy operator by:

$$\begin{aligned} E^\alpha : H^\alpha(\mathbb{R}) \times H^{\alpha+1}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\zeta, u) &\longmapsto \sqrt{|\zeta|_{H^\alpha}^2 + |u|_{H^{\alpha+1}}^2}. \end{aligned}$$

Since  $s > 1/2 + \max(\lambda, 1)$ ,  $H^{\alpha_l^m}$  is an algebra for all  $l = 0, \dots, m$ . To simplify the notation in this Section 3.1, we denote

$$\alpha := \alpha_l^m.$$

Below, we use the energy estimate to show that the energy operator  $E^\alpha(\zeta_l, u_l)$  is bounded in the time interval  $[0, T]$ . We apply the operator  $\Lambda^\alpha$  to (3.4)<sub>1</sub> and (3.4)<sub>2</sub>, then multiply these two equations by



$\Lambda^\alpha \zeta_l$  and  $\Lambda^\alpha u_l$  respectively.

We integrate this new system with respect to  $x$  and we sum the two equations. We find

$$\begin{aligned} \partial_t(E^\alpha(\zeta_l, u_l)^2) &= -(\Lambda^\alpha(u_l)_x, \Lambda^\alpha \zeta_l)_{L^2} - (\Lambda^\alpha(u \zeta_l)_x, \Lambda^\alpha \zeta_l)_{L^2} - (\Lambda^\alpha(u_l \zeta)_x, \Lambda^\alpha \zeta_l)_{L^2} \\ &\quad - (\Lambda^\alpha(\zeta_l)_x, \Lambda^\alpha u_l)_{L^2} - (\Lambda^\alpha(u u_l)_x, \Lambda^\alpha u_l)_{L^2} + (\Lambda^\alpha f_{1,l}, \Lambda^\alpha \zeta_l)_{L^2} + (\Lambda^\alpha f_{2,l}, \Lambda^\alpha u_l)_{L^2} \\ &= \sum_{j=1}^7 \sigma_j, \end{aligned} \quad (3.7)$$

where the  $\sigma_j$  are the  $j$ -th monomial in the equation above. The functions  $f_{1,l}$  and  $f_{2,l}$  are defined in (3.5).

We now turn to estimating the different terms of the previous identity:

Estimate of  $\sigma_1$  and  $\sigma_4$ : We use (2.3) and get

$$\sigma_1 + \sigma_4 = (\Lambda^\alpha(u_l)_x, \Lambda^\alpha \zeta_l)_{L^2} + (\Lambda^\alpha(\zeta_l)_x, \Lambda^\alpha u_l)_{L^2} = 0. \quad (3.8)$$

Estimate of  $\sigma_2$ : We use here the Cauchy-Schwarz inequality and the definition of the commutator (2.4). One has

$$\begin{aligned} |\sigma_2| &= |(\Lambda^\alpha(u \cdot \zeta_l)_x, \Lambda^\alpha \zeta_l)_{L^2}| = |(\Lambda^\alpha(u \cdot \zeta_l), \Lambda^\alpha(\zeta_l)_x)_{L^2}| \\ &\leq |(u \cdot \Lambda^\alpha \zeta_l, (\Lambda^\alpha \zeta_l)_x)_{L^2}| + |([\Lambda^\alpha, u] \zeta_l, (\Lambda^\alpha \zeta_l)_x)_{L^2}| \\ &= \frac{1}{2} |(u_x, |\Lambda^\alpha \zeta_l|^2)_{L^2}| + |(\partial_x([\Lambda^\alpha, u] \zeta_l), \Lambda^\alpha \zeta_l)_{L^2}| \\ &= \frac{1}{2} |(u_x \cdot \Lambda^\alpha \zeta_l, \Lambda^\alpha \zeta_l)_{L^2}| + |(\partial_x([\Lambda^\alpha, u] \zeta_l), \Lambda^\alpha \zeta_l)_{L^2}| \\ &\leq \frac{1}{2} |u_x|_{L^\infty} \cdot |\zeta_l|_{H^\alpha}^2 + |(\partial_x([\Lambda^\alpha, u] \zeta_l), \Lambda^\alpha \zeta_l)_{L^2}|. \end{aligned}$$

For the first term of the above inequality, we use that  $H^\alpha(\mathbb{R}) \subset L^\infty(\mathbb{R})$ , we get

$$\frac{1}{2} |u_x|_{L^\infty} |\zeta_l|_{H^\alpha}^2 \lesssim \frac{1}{2} |u_x|_{H^\alpha} |\zeta_l|_{H^\alpha}^2. \quad (3.9)$$

Then, for the second term by using the Cauchy-Schwarz inequality and (2.6), we get

$$|(\partial_x([\Lambda^\alpha, u] \zeta_l), \Lambda^\alpha \zeta_l)_{L^2}| \lesssim (|[\Lambda^\alpha, u_x] \zeta_l|_{L^2} + |[\Lambda^\alpha, u](\zeta_l)_x|_{L^2}) |\Lambda^\alpha \zeta_l|_{L^2},$$

We use below (2.7) for the first term ( $r = \alpha, b = \alpha - 1$ ) and for the second term ( $r = \alpha, b = \alpha$ ). We obtain

$$\begin{aligned} (|[\Lambda^\alpha, u_x] \zeta_l|_{L^2} + |[\Lambda^\alpha, u](\zeta_l)_x|_{L^2}) |\Lambda^\alpha \zeta_l|_{L^2} &\lesssim (|u_{xx}|_{H^{\alpha-1}} |\zeta_l|_{H^{\alpha-1}} + |u_x|_{H^\alpha} |(\zeta_l)_x|_{H^{\alpha-1}}) |\zeta_l|_{H^\alpha} \\ &\lesssim 2 |u_x|_{H^\alpha} |\zeta_l|_{H^\alpha}^2. \end{aligned} \quad (3.10)$$

By (3.9) and (3.10), we can state

$$\begin{aligned} |\sigma_2| &= |(\Lambda^\alpha(u \zeta_l)_x, \Lambda^\alpha \zeta_l)_{L^2}| \lesssim \frac{5}{2} |u_x|_{H^\alpha} |\zeta_l|_{H^\alpha}^2 \\ &\lesssim \frac{5}{2} |u_x|_{H^\alpha} E^\alpha(\zeta_l, u_l)^2. \end{aligned} \quad (3.11)$$

Estimate of  $\sigma_3$ : We use here the Cauchy-Schwarz inequality. We get

$$\begin{aligned} |\sigma_3| &= |(\Lambda^\alpha(u_l \zeta)_x, \Lambda^\alpha \zeta_l)_{L^2}| \leq |u_l \zeta|_{H^{\alpha+1}} |\zeta_l|_{H^\alpha} \\ &\leq |u_l|_{H^{\alpha+1}} |\zeta|_{H^{\alpha+1}} |\zeta_l|_{H^\alpha} \\ &\lesssim |\zeta|_{H^{\alpha+1}} E^\alpha(\zeta_l, u_l)^2. \end{aligned} \quad (3.12)$$

Estimate of  $\sigma_5$ : We use (2.1) such that  $r = \alpha$  and  $r_1 = r_2 = \alpha + 1$  and we get

$$\begin{aligned} |\sigma_5| &= |(\Lambda^\alpha(u u_l)_x, \Lambda^\alpha u_l)_{L^2}| \leq |\Lambda^\alpha(u u_l)|_{L^2} |\Lambda^\alpha(u_l)_x|_{L^2} \\ &\lesssim |u|_{H^{\alpha+1}} |u_l|_{H^{\alpha+1}}^2 \\ &\lesssim |u|_{H^{\alpha+1}} E^\alpha(\zeta_l, u_l)^2. \end{aligned} \quad (3.13)$$

Estimate of  $\sigma_6$ : We use the triangle inequality for  $f_{1,l}$  defined in (3.5). We find

$$\begin{aligned} |\sigma_6| &= |(\Lambda^\alpha f_{1,l}, \Lambda^\alpha \zeta_l)_{L^2}| \leq \sum_{i=1}^{l-1} |(\Lambda^\alpha(\zeta_i u_{l-i})_x, \Lambda^\alpha \zeta_l)_{L^2}| + |(\Lambda^\alpha g_\lambda[\zeta_{l-1}], \Lambda^\alpha \zeta_l)_{L^2}|. \\ &=: |\sigma_{6.1}| + |\sigma_{6.2}| \end{aligned}$$

1. To estimate  $\sigma_{6.1}$ , we use (2.1) such that  $r = r_1 = r_2 = \alpha + 1$ , and get

$$\begin{aligned} |\sigma_{6.1}| &= \sum_{i=1}^{l-1} |(\Lambda^\alpha(\zeta_i u_{l-i})_x; \Lambda^\alpha \zeta_l)_{L^2}| \lesssim \sum_{i=1}^{l-1} |\zeta_i u_{l-i}|_{H^{\alpha+1}} |\zeta_l|_{H^\alpha}, \\ &\lesssim \sum_{i=1}^{l-1} |\zeta_i|_{H^{\alpha+1}} |u_{l-i}|_{H^{\alpha+1}} |\zeta_l|_{H^\alpha}. \end{aligned} \quad (3.14)$$

2. To estimate  $\sigma_{6.2}$ , we use the Cauchy-Schwarz inequality. We have

$$\begin{aligned} |\sigma_{6.2}| &= |(\Lambda^\alpha g_\lambda[\zeta_{l-1}]; \Lambda^\alpha \zeta_l)_{L^2}| = \left| \int_{\mathbb{R}} \Lambda^\alpha \widehat{g_\lambda[\zeta_{l-1}]} \overline{\Lambda^\alpha \zeta_l} \, dk \right| \\ &= \left| \int_{\mathbb{R}} (1+k^2)^{\alpha/2} |k|^\lambda \widehat{\zeta_{l-1}}(k) \overline{\Lambda^\alpha \zeta_l} \, dk \right| \\ &\leq \left( \int_{\mathbb{R}} (1+k^2)^{\alpha+\lambda} |\widehat{\zeta_{l-1}}|^2(k) \, dk \right)^{\frac{1}{2}} |\zeta_l|_{H^\alpha} \\ &= |\zeta_{l-1}|_{H^{\alpha+\lambda}} |\zeta_l|_{H^\alpha}. \end{aligned} \quad (3.15)$$

By (3.14) and (3.15), we can state

$$|\sigma_6| = |(\Lambda^\alpha f_{1,l}, \Lambda^\alpha \zeta_l)_{L^2}| \lesssim \left[ \sum_{i=1}^{l-1} |\zeta_i|_{H^{\alpha+1}} |u_{l-i}|_{H^{\alpha+1}} + |\zeta_{l-1}|_{H^{\alpha+\lambda}} \right] E^\alpha(\zeta_l, u_l). \quad (3.16)$$

Estimate of  $\sigma_7$ : We use (2.1) for  $f_l^2$  given by (3.5), and get

$$\begin{aligned} |\sigma_7| &= |(\Lambda^\alpha f_{2,l}, \Lambda^\alpha u_l)_{L^2}| \lesssim \sum_{i=1}^{l-1} |\Lambda^\alpha(u_i(u_{l-i})_x)|_{L^2} |u_l|_{H^\alpha} \\ &\lesssim \sum_{i=1}^{l-1} |u_i|_{H^\alpha} |u_{l-i}|_{H^{\alpha+1}} E^\alpha(\zeta_l, u_l). \end{aligned} \quad (3.17)$$

Now, we put the estimates (3.8), (3.11), (3.12), (3.13), (3.16) and (3.17) in (3.7). We get

$$\begin{aligned} \partial_t(E^\alpha(\zeta_l, u_l)^2) &\lesssim \left[ \frac{5}{2}|u_x|_{H^\alpha} + |\zeta|_{H^{\alpha+1}} + |u|_{H^{\alpha+1}} \right] E^\alpha(\zeta_l, u_l)^2 \\ &\quad + \left[ |\zeta_{l-1}|_{H^{\alpha+\lambda}} + \sum_{i=1}^{l-1} \left( |\zeta_i|_{H^{\alpha+1}} |u_{l-i}|_{H^{\alpha+1}} + |u_i|_{H^\alpha} |u_{l-i}|_{H^{\alpha+1}} \right) \right] E^\alpha(\zeta_l, u_l), \\ &=: A(t)E^\alpha(\zeta_l, u_l)^2 + B(t)E^\alpha(\zeta_l, u_l). \end{aligned} \quad (3.18)$$

We use (3.18) to prove that the operator  $E^\alpha(\zeta_l, u_l)$  is bounded in the time interval  $[0, T]$ . For all  $T > 0$ , we denote  $A := \sup_{t \in [0, T]} A(t)$ . Then

$$\partial_t(E^\alpha(\zeta_l, u_l)^2) \leq AE^\alpha(\zeta_l, u_l)^2 + B(t)E^\alpha(\zeta_l, u_l).$$

Now, we multiply by  $e^{-At}$  and get

$$\partial_t(e^{-At}E^\alpha(\zeta_l, u_l)^2) \leq e^{-At}B(t)E^\alpha(\zeta_l, u_l).$$

We get

$$E^\alpha(\zeta_l, u_l) \lesssim e^{At/2}E^\alpha(\zeta_{l0}, u_{l0}) + \frac{1}{2} \int_0^t e^{A(t-r)/2} B(r) dr. \quad (3.19)$$

So, we completed the proof of Theorem 3.2. □

### 3.2 Smallness of the remaining term in the expansion

Below, we prove that the solutions  $(\zeta_l, u_l)$ ,  $l = 1, \dots, m$  given by Theorem 3.2 in Section 3.1 and the solution  $(\zeta, u)$  of the system (1.1) are actually the coefficients of the asymptotic expansion of the solution  $(\zeta_\epsilon, u_\epsilon)$  with respect to  $\epsilon$ . To that purpose, we subtract the expansion from the regularized solution and prove that the remaining term is small in functional spaces.

In the following, we use the notation  $U_\epsilon = (\zeta_\epsilon, u_\epsilon)$  for the solution of system (1.2),  $U_l = (\zeta_l, u_l)$  where  $l = 1, \dots, m$  for the solutions given by Theorem 3.2, and  $U_0 := U = (\zeta, u)$  the solution of system (1.1). Let  $V_l$  be defined by induction by the formulas:

$$\begin{cases} V_0 = U_\epsilon, \\ V_{l+1} = V_l - \epsilon^l U_l \quad \text{for } l = 0, \dots, m. \end{cases} \quad (3.20)$$

This definition is designed so as to have the following expansion with the remaining  $V_{l+1}$ :

$$U_\epsilon = U + \sum_{i=1}^l \epsilon^i U_i + V_{l+1}. \quad (3.21)$$

We denote here  $X_l^m := H^{\alpha_{l-1}^m}(\mathbb{R}) \times H^{\alpha_{l-1}^m+1}(\mathbb{R})$ , where  $\alpha_l^m$  was defined in (3.2).

**Remark 5** The injection mapping  $X_l^m \subset X_{l+1}^m$  is continuous. Moreover  $X_l^m$  is still an algebra up to  $l = m + 2$  thanks to the assumption (3.1).

In the following we will use the following Lemma, so as to state that the rest of the expansion tends to zero in functional spaces when  $\epsilon$  tends to zero.

**Lemma 3.1** Let  $\epsilon \geq 0$ . Then for  $l = 0, \dots, m$ ,  $V_{l+1}$  defined by (3.20) belongs to  $X_{l+1}^m$ . Moreover, as  $\epsilon$  tends to zero,  $V_{l+1}$  tends to zero in  $X_{l+1}^m$ .

**Proof:** (by induction)

For  $l = 0$ , it is clear that  $V_1 = U_\epsilon - U \in X_1^m$ . Then, we use the convergence (1.3) that is proven in Theorem 1.1 in [9]. We get  $V_1 = U_\epsilon - U$  goes to zero in  $X_1^m$  when  $\epsilon \rightarrow 0$ .

For  $l = 1$ , by definition  $V_2 = V_1 - \epsilon U_1$ . We just proved that  $V_1$  tends to zero in  $X_1^m \subset X_2^m$ , and  $U_1$  defined by Theorem 3.2 (for  $l = 1$ ) then bounded in  $X_2^m$ . Therefore,  $V_2$  tends to zero in  $X_2^m$  when  $\epsilon \rightarrow 0$ .

Now, by induction step, we suppose that the property of Lemma 3.1 is true for  $l$ . Hence  $V_{l+1} = V_l - \epsilon^l U_l \in X_{l+1}^m$  and  $V_{l+1}$  tends to zero in  $X_{l+1}^m$  when  $\epsilon \rightarrow 0$  (the same interpretation as the case  $l = 1$ ). This completes the proof of Lemma 3.1. □

The expression (3.21) defines the asymptotic expansion of  $U_\epsilon$  where  $V_{l+1}$  is the rest of the expansion. To complete the proof of Theorem 3.1, we must prove that: for  $l = 0, \dots, m$

$$V_{l+1} = o(\epsilon^l). \quad (3.22)$$

For that purpose, we expand  $U_\epsilon$  up to order  $l$  in (1.2) and we write the system satisfied by  $V_{l+1} =: (p, q)$ . After simplifications of the order 0 to  $l$  terms, thanks to Theorem 3.2, we get this nonlinear system:

$$\begin{cases} p_t + q_x + \sum_{i=0}^l [\epsilon^i (\zeta_i q)_x + \epsilon^i (u_i p)_x] + (pq)_x + \epsilon g_\lambda[p] = F_{1,l+1}, \\ q_t - q_{txx} + p_x + \sum_{i=0}^l \epsilon^i (u_i q)_x + qq_x = F_{2,l+1}, \end{cases} \quad (3.23)$$

where,  $0 \leq l \leq m$  and:

$$F_{1,l+1} = - \sum_{k=1}^l \sum_{i=1}^k \epsilon^{k+l+1-i} (\zeta_k u_{l+1-i})_x - \epsilon^{l+1} g_\lambda[\zeta_l], \quad F_{2,l+1} = - \sum_{k=1}^l \sum_{i=1}^k \epsilon^{k+l+1-i} u_k (u_{l+1-i})_x,$$

and vanishing initial conditions (if  $l \geq 1$ ). By (3.20), the solution  $V_{l+1}$  of the system (3.23) belongs to  $X_{l+1}^m$ .

To simplify the notation here, let  $D_{l+1}$  be the nonlinear partial differential operator defined by the system (3.23), i.e.

$$D_{l+1}(V_{l+1}) = (F_{1,l+1}, F_{2,l+1}). \quad (3.24)$$

Now, we need two technical lemmas to complete the proof of the Theorem 3.1. We will prove first that

the right hand side of (3.24) is  $o(\epsilon^l)$  in adequate norms. Then, we will prove that the solution  $V_{l+1}$  to (3.24) depends continuously on the right hand side. The conclusion will be obvious then.

**Lemma 3.2** For  $0 \leq l \leq m$ , we have

$$(F_{1,l+1}, F_{2,l+1}) = o(\epsilon^l) \quad \text{in} \quad L_t^\infty X_{l+2}^m.$$

**Remark 6** By using (3.1), we get that  $X_{m+2}^m$  is still an algebra.

**Proof:**

We compute the norm of  $(F_{1,l+1}, F_{2,l+1})$  in  $L_t^\infty X_{l+2}^m$  where  $X_{l+2}^m = H^{\alpha_{l+1}^m}(\mathbb{R}) \times H^{\alpha_{l+1}^m+1}(\mathbb{R})$ . To simplify the notation in this Section 3.2, we take  $\tilde{\alpha} := \alpha_{l+1}^m$ .

**Remark 7** By assumption, we have  $\zeta_k \in H^{\alpha_k^m}$ . But, if  $k \leq l$

$$\begin{aligned} \tilde{\alpha} + 1 &= s + (m - l) \max(\lambda, 1) - \max(\lambda, 1) + 1 \\ &\leq s + (m - k) \max(\lambda, 1), \quad \text{because } k \leq l, \\ &= \alpha_k^m, \end{aligned}$$

then  $\zeta_k \in H^{\alpha_k^m} \subset H^{\tilde{\alpha}+1}$ . So  $|\zeta_k|_{H^{\tilde{\alpha}+1}} < \infty$ . The same for the existence of  $|\zeta_l|_{H^{\tilde{\alpha}+\lambda}}$  and  $|u_{l+1-i}|_{H^{\tilde{\alpha}+2}}$ .

- For  $F_{1,l+1}$ , we use (2.1). We get:

$$\begin{aligned} |F_{1,l+1}|_{L_t^\infty H^{\tilde{\alpha}}} &= |\Lambda^{\tilde{\alpha}} F_{1,l+1}|_{L_t^\infty L_x^2} \lesssim \epsilon^{l+1} \cdot \left[ \sum_{k=1}^l \sum_{i=1}^k \epsilon^{k-i} |\zeta_k|_{L_t^\infty H^{\tilde{\alpha}+1}} |u_{l+1-i}|_{L_t^\infty H^{\tilde{\alpha}+1}} + |\zeta_l|_{L_t^\infty H^{\tilde{\alpha}+\lambda}} \right] \\ &\lesssim \epsilon^{l+1} \cdot \left[ \sum_{k=1}^l \sum_{i=1}^k \epsilon^{k-i} |\zeta_k|_{L_t^\infty H^{\alpha_k^m}} |u_{l+1-i}|_{L_t^\infty H^{\alpha_{l+1-i}^m}} + |\zeta_l|_{L_t^\infty H^{\alpha_l^m}} \right]. \end{aligned}$$

- For  $F_{2,l+1}$ , we use (2.1). We get:

$$\begin{aligned} |F_{2,l+1}|_{L_t^\infty H^{\tilde{\alpha}+1}} &= |\Lambda^{\tilde{\alpha}+1} F_{2,l+1}|_{L_t^\infty L_x^2} \lesssim \epsilon^{l+1} \cdot \left[ \sum_{k=1}^l \sum_{i=1}^k \epsilon^{k-i} |u_k|_{L_t^\infty H^{\tilde{\alpha}+1}} |u_{l+1-i}|_{L_t^\infty H^{\tilde{\alpha}+2}} \right] \\ &\lesssim \epsilon^{l+1} \cdot \left[ \sum_{k=1}^l \sum_{i=1}^k \epsilon^{k-i} |u_k|_{L_t^\infty H^{\alpha_k^m+1}} |u_{l+1-i}|_{L_t^\infty H^{\alpha_{l+1-i}^m+1}} \right]. \end{aligned}$$

We conclude that

$$(F_{1,l+1}, F_{2,l+1}) = o(\epsilon^l) \quad \text{in} \quad L_t^\infty X_{l+2}^m.$$

□

The function  $V_{l+1} \in X_{l+1}^m \subset X_{l+2}^m$  satisfies (3.23). We have the following Lemma:

**Lemma 3.3** For  $(F_{1,l+1}, F_{2,l+1}) \in L_t^\infty X_{l+2}^m$ , we have

$$|V_{l+1}|_{L_t^\infty X_{l+2}^m} \leq C |(F_{1,l+1}, F_{2,l+1})|_{L_t^\infty X_{l+2}^m}.$$

**Proof:**

We apply the operator  $\Lambda^{\tilde{\alpha}}$  to (3.23)<sub>1</sub> and (3.23)<sub>2</sub>, then multiply these two equations by  $\Lambda^{\tilde{\alpha}}p$  and  $\Lambda^{\tilde{\alpha}}q$  respectively. We integrate this new system with respect to  $x$  and we sum the two equations. We find

$$\begin{aligned}
& \partial_t (|p|_{H^{\tilde{\alpha}}}^2 + |q|_{H^{\tilde{\alpha}+1}}^2) + \epsilon (\Lambda^{\tilde{\alpha}}g_\lambda[p], \Lambda^{\tilde{\alpha}}p)_{L^2} \\
&= -(\Lambda^{\tilde{\alpha}}(pq)_x, \Lambda^{\tilde{\alpha}}p)_{L^2} - (\Lambda^{\tilde{\alpha}}(qq_x), \Lambda^{\tilde{\alpha}}q)_{L^2} \\
&\quad - \sum_{i=0}^l \epsilon^i \left( (\Lambda^{\tilde{\alpha}}(\zeta_i q)_x, \Lambda^{\tilde{\alpha}}p)_{L^2} + (\Lambda^{\tilde{\alpha}}(u_i p)_x, \Lambda^{\tilde{\alpha}}p)_{L^2} + (\Lambda^{\tilde{\alpha}}(u_i q)_x, \Lambda^{\tilde{\alpha}}q)_{L^2} \right) \\
&\quad + (\Lambda^{\tilde{\alpha}}F_{1,l+1}, \Lambda^{\tilde{\alpha}}p)_{L^2} + (\Lambda^{\tilde{\alpha}}F_{2,l+1}, \Lambda^{\tilde{\alpha}}q)_{L^2} \\
&= \sum_{j=1}^7 \tau_j,
\end{aligned} \tag{3.25}$$

where the  $\tau_j$  are the  $j$ -th monomial up to these numbers in the equation. In [9], the authors prove that

$$(\Lambda^{\tilde{\alpha}}g_\lambda[p], \Lambda^{\tilde{\alpha}}p)_{L^2} \geq |p_x|_{H^{\tilde{\alpha}-1+\lambda/2}}^2 \geq 0.$$

We now turn to estimating the different terms of (3.25). In the following, we will do the same calculations that we did in the proof of the Energy estimate (3.6):

Estimate of  $\tau_1$ : we do the same estimate as  $\sigma_2$ :

$$|\tau_1| = |(\Lambda^{\tilde{\alpha}}(pq)_x, \Lambda^{\tilde{\alpha}}p)_{L^2}| \lesssim \frac{5}{2} |q|_{H^{\tilde{\alpha}+1}} |p|_{H^{\tilde{\alpha}}}^2. \tag{3.26}$$

Estimate of  $\tau_2$ : we use here the Cauchy-Schwarz inequality. We get

$$|\tau_2| = |(\Lambda^{\tilde{\alpha}}(qq_x), \Lambda^{\tilde{\alpha}}q)_{L^2}| \leq |q|_{L^\infty} |q|_{H^{\tilde{\alpha}+1}}^2. \tag{3.27}$$

Estimate of  $\tau_3$ : by the same estimate as for  $\sigma_{6.1}$ , we get

$$\begin{aligned}
|\tau_3| &= \left| \sum_{i=0}^l \epsilon^i (\Lambda^{\tilde{\alpha}}(\zeta_i q)_x, \Lambda^{\tilde{\alpha}}p)_{L^2} \right| \leq \sum_{i=0}^l \epsilon^i |\zeta_i q|_{H^{\tilde{\alpha}+1}} |p|_{H^{\tilde{\alpha}}} \\
&\leq \sum_{i=0}^l \epsilon^i |\zeta_i|_{H^{\tilde{\alpha}+1}} |q|_{H^{\tilde{\alpha}+1}} |p|_{H^{\tilde{\alpha}}} \\
&\leq \sum_{i=0}^l \epsilon^i |\zeta_i|_{H^{\alpha_i^m}} |q|_{H^{\tilde{\alpha}+1}} |p|_{H^{\tilde{\alpha}}}.
\end{aligned} \tag{3.28}$$

Estimate of  $\tau_4$ : we do the same step as for  $\tau_1$ . We get

$$\begin{aligned}
|\tau_4| &= \left| \sum_{i=0}^l \epsilon^i (\Lambda^{\tilde{\alpha}}(u_i p)_x, \Lambda^{\tilde{\alpha}}p)_{L^2} \right| \lesssim \frac{5}{2} \sum_{i=0}^l \epsilon^i |u_i|_{H^{\tilde{\alpha}+1}} |p|_{H^{\tilde{\alpha}}}^2 \\
&\lesssim \frac{5}{2} \sum_{i=0}^l \epsilon^i |u_i|_{H^{\alpha_i^m+1}} |p|_{H^{\tilde{\alpha}}}^2.
\end{aligned} \tag{3.29}$$

Estimate of  $\tau_5$ : we do the same step as for  $\tau_3$ . We have

$$\begin{aligned}
|\tau_5| &= \left| \sum_{i=0}^l \epsilon^i (\Lambda^{\tilde{\alpha}}(u_i q)_x, \Lambda^{\tilde{\alpha}} q)_{L^2} \right| \leq \sum_{i=0}^l \epsilon^i |u_i q|_{H^{\tilde{\alpha}+1}} |q|_{H^{\tilde{\alpha}}} \\
&\leq \sum_{i=0}^l \epsilon_i |u_i|_{H^{\tilde{\alpha}+1}} |q|_{H^{\tilde{\alpha}+1}}^2 \\
&\leq \sum_{i=0}^l \epsilon^i |u_i|_{H^{\alpha_i^{m+1}}} |q|_{H^{\tilde{\alpha}+1}}^2.
\end{aligned} \tag{3.30}$$

Estimate of  $\tau_6$ : we use here the Cauchy-Schwarz inequality. We get

$$|\tau_6| = |(\Lambda^{\tilde{\alpha}} F_{1,l+1}, \Lambda^{\tilde{\alpha}} p)_{L^2}| \leq |F_{1,l+1}|_{H^{\tilde{\alpha}}} |p|_{H^{\tilde{\alpha}}}. \tag{3.31}$$

Estimate of  $\tau_7$ : we use the Cauchy-Schwarz inequality. We get

$$|\tau_7| = |(\Lambda^{\tilde{\alpha}} F_{2,l+1}, \Lambda^{\tilde{\alpha}} q)_{L^2}| \leq |F_{2,l+1}|_{H^{\tilde{\alpha}+1}} |q|_{H^{\tilde{\alpha}+1}}. \tag{3.32}$$

Now, we put the estimates (3.26), (3.27), (3.28), (3.29), (3.30), (3.31) and (3.32) in (3.25). We get

$$\begin{aligned}
\partial_t \left( |p|_{H^{\tilde{\alpha}}}^2 + |q|_{H^{\tilde{\alpha}+1}}^2 \right) + \epsilon |p_x|_{H^{\tilde{\alpha}-1+\lambda/2}}^2 &\lesssim \left[ |p|_{H^{\tilde{\alpha}}} |q|_{H^{\tilde{\alpha}+1}} + |q|_{L_x^\infty} |q|_{H^{\tilde{\alpha}+1}} \right. \\
&\quad + \sum_{i=0}^l \epsilon^i |\zeta_i|_{H^{\alpha_i^m}} |q|_{H^{\tilde{\alpha}+1}} + \sum_{i=0}^l \epsilon^i |u_i|_{H^{\alpha_i^{m+1}}} |p|_{H^{\tilde{\alpha}}} \\
&\quad + \sum_{i=0}^l \epsilon^i |u_i|_{H^{\alpha_i^{m+1}}} |q|_{H^{\tilde{\alpha}+1}} \\
&\quad \left. + |F_{1,l+1}|_{H^{\tilde{\alpha}}} + |F_{2,l+1}|_{H^{\tilde{\alpha}+1}} \right] \left( |p|_{H^{\tilde{\alpha}}} + |q|_{H^{\tilde{\alpha}+1}} \right).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\partial_t \left( E^{\tilde{\alpha}}(p, q) \right)^2 &\lesssim \left( |F_{1,l+1}|_{H^{\tilde{\alpha}}} + |F_{2,l+1}|_{H^{\tilde{\alpha}+1}} \right) |E^{\tilde{\alpha}}(p, q)| \\
&\quad + \left[ |q|_{H^{\tilde{\alpha}+1}} + |q|_{L_x^\infty} + \sum_{i=0}^l \epsilon^i \left( |\zeta_i|_{H^{\alpha_i^m}} + |u_i|_{H^{\alpha_i^{m+1}}} \right) \right] \left( E^{\tilde{\alpha}}(p, q) \right)^2.
\end{aligned}$$

To simplify the notation, we take

$$\Gamma(t) := |q|_{H^{\tilde{\alpha}+1}} + |q|_{L_x^\infty} + \sum_{i=0}^l \epsilon^i \left( |\zeta_i|_{H^{\alpha_i^m}} + |u_i|_{H^{\alpha_i^{m+1}}} \right).$$

- If  $E^{\tilde{\alpha}}(p, q) = 0$ , then  $p = q = 0$  and this completes the proof of Lemma 3.3 .
- If  $E^{\tilde{\alpha}}(p, q) > 0$ , then we divide by  $E^{\tilde{\alpha}}(p, q)$ . We get

$$\partial_t E^{\tilde{\alpha}}(p, q) \lesssim \Gamma(t) E^{\tilde{\alpha}}(p, q) + \left( |F_{1,l+1}|_{H^{\tilde{\alpha}}} + |F_{2,l+1}|_{H^{\tilde{\alpha}+1}} \right).$$

Then, we have

$$\begin{aligned} E^{\tilde{\alpha}}(p, q) &\lesssim \left[ \int_0^t \left( |F_{1,l+1}|_{H^{\tilde{\alpha}}} + |F_{2,l+1}|_{H^{\tilde{\alpha}+1}} \right) \exp\left(-\int_0^\tau \Gamma(s) ds\right) d\tau \right] \exp\left(\int_0^t \Gamma(s) ds\right) \\ &\lesssim \left( |F_{1,l+1}|_{L_t^\infty H^{\tilde{\alpha}}} + |F_{2,l+1}|_{L_t^\infty H^{\tilde{\alpha}+1}} \right) \left[ \int_0^t \exp\left(-\int_0^\tau \Gamma(s) ds\right) d\tau \right] \exp\left(\int_0^t \Gamma(s) ds\right). \end{aligned}$$

Therefore,

$$|V_{l+1}|_{L_t^\infty X_{l+2}^m} \lesssim \left( |F_{1,l+1}|_{L_t^\infty H^{\tilde{\alpha}}} + |F_{2,l+1}|_{L_t^\infty H^{\tilde{\alpha}+1}} \right).$$

So, we completed the proof of Lemma 3.3. □

At this stage, we proved the existence of  $m$  vectors  $(\zeta_l, u_l)_{l=1, \dots, m}$  in Theorem 3.2. Then, we proved that the rest of the asymptotic expansion  $V_{l+1}$  is  $o(\epsilon^l)$  in Lemmas 3.2 and 3.3. Now, we will use all these results to complete the proof of Theorem 3.1.

**Proof of Theorem 3.1:** By definition (3.20) of  $V_m$ , we deduce

$$U_\epsilon = U + \sum_{l=1}^m \epsilon^l U_l + V_{m+1}.$$

Applying Lemma 3.2 to  $l = m + 1$ , gives us

$$(F_{1,m+1}, F_{2,m+1}) = o(\epsilon^m) \quad \text{in } L_t^\infty X_{m+2}^m(\mathbb{R}).$$

Using Lemma 3.3, we get

$$|V_{m+1}|_{L_t^\infty X_{m+2}^m} \lesssim |(F_{1,m+1}, F_{2,m+1})|_{L_t^\infty X_{m+2}^m} = o(\epsilon^m).$$

Therefore, the solution  $U_\epsilon$  to the Cauchy problem associated with (1.2) has, in  $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ , an asymptotic expansion up to order  $m$ ,

$$U_\epsilon = U + \sum_{l=1}^m \epsilon^l U_l + o(\epsilon^m),$$

and the proof is complete. □

## 4 Numerical simulations

Below, we give the numerical justifications and results that transpose in numerics our theoretical Theorem 3.1. In the first part, we define the numerical scheme and we verify its convergence to the expected solution. In the second part, we verify numerically the result of [9] that the solution of the regularized Boussinesq system (1.2) tends to the solution of the Boussinesq system (1.1) when  $\epsilon$  tends to zero. Then in the third, we check numerically the existence of an asymptotic expansion of the solution  $(\zeta_\epsilon, u_\epsilon)$  of the system (1.2) with respect to  $\epsilon$  up to the order 2 as stated in our general Theorem 3.1 in the general case  $m$ .



## 4.1 Numerical scheme

We start with the discretization in time and space for the system (1.2) :

### The time discretization

We use a Sanz-Serna scheme for which  $dt$  is the time step. The system (1.2) is discretized in time in:

$$\begin{cases} \frac{\zeta^{n+1} - \zeta^n}{dt} + \partial_x \left( \frac{u^{n+1} + u^n}{2} \right) + \partial_x \left( \frac{(\zeta^{n+1} + \zeta^n)(u^{n+1} + u^n)}{4} \right) + \epsilon g_\lambda \left[ \frac{\zeta^{n+1} + \zeta^n}{2} \right] = 0, \\ (1 - \partial_{xx}) \cdot \frac{u^{n+1} - u^n}{dt} + \partial_x \left( \frac{\zeta^{n+1} + \zeta^n}{2} \right) + \frac{1}{2} \partial_x \left( \frac{u^{n+1} + u^n}{2} \right)^2 = 0. \end{cases} \quad (4.1)$$

### The space discretization

To simplify the notation,  $k$  denotes the value of  $\frac{2\pi k}{L}$ . We apply the Fourier transform in space to the system (4.1). We get

$$\begin{cases} \hat{\zeta}_k^{n+1} = \left[ \left( 1 - \frac{dt}{2} \epsilon |k|^\lambda \right) \hat{\zeta}_k^n - \frac{1}{2} ik dt \cdot (\hat{u}_k^{n+1} + \hat{u}_k^n) \right. \\ \quad \left. - \frac{1}{4} ik dt \cdot \mathcal{F} \left( (\zeta^{n+1} + \zeta^n)(u^{n+1} + u^n) \right) \right] \left( 1 + \frac{dt}{2} \epsilon |k|^\lambda \right)^{-1}, \\ \hat{u}_k^{n+1} = \left[ (1 + k^2) \hat{u}_k^n - \frac{1}{2} ik dt \cdot (\hat{\zeta}_k^{n+1} + \hat{\zeta}_k^n) - \frac{1}{8} ik dt \cdot \mathcal{F} \left( (u^{n+1} + u^n)^2 \right) \right] (1 + k^2)^{-1}. \end{cases} \quad (4.2)$$

This scheme is implicit, so we will apply a fixed point method. To that end, we use the standard Picard iterate to solve the fixed point at each step. The system (4.2) can be written as :

$$\begin{cases} \hat{\zeta}^{n+1} = \Theta_1(\hat{\zeta}^{n+1}, \hat{\zeta}^n, \hat{u}^{n+1}, \hat{u}^n), \\ \hat{u}^{n+1} = \Theta_2(\hat{\zeta}^{n+1}, \hat{\zeta}^n, \hat{u}^{n+1}, \hat{u}^n), \end{cases} \quad (4.3)$$

where  $\Theta_1$  and  $\Theta_2$  are the right hand side of System (4.2).

We solve this system by the following algorithm:

---

**Algorithm** : Picard iterate

---

$$v_1^0 = \zeta^n, v_2^0 = u^n, m = 0$$

**While**  $\|\Theta_1(\hat{v}_1^m, \hat{\zeta}^n, \hat{v}_2^m, \hat{u}^n) - \hat{v}_1^m\| > \theta$  &  $\|\Theta_2(\hat{v}_1^m, \hat{\zeta}^n, \hat{v}_2^m, \hat{u}^n) - \hat{v}_2^m\| > \theta$  **do**

$$\hat{v}_1^{m+1} = \Theta_1(\hat{v}_1^m, \hat{\zeta}^n, \hat{v}_2^m, \hat{u}^n)$$

$$\hat{v}_2^{m+1} = \Theta_2(\hat{v}_1^m, \hat{\zeta}^n, \hat{v}_2^m, \hat{u}^n)$$

m=m+1

**end While**

$$\zeta^{n+1} = v_1^m \text{ and } u^{n+1} = v_2^m$$


---

Here  $\theta > 0$  is a small fixed parameter. We choose  $\theta = 10^{-13}$ . We will use this algorithm for the whole

article.

**Stability of the Picard iterate:**

One may wonder whether the substep of Picard fixed point, which relies on a false time remains stable. To that end, we follow what is done in [16]. In this article, the authors use the classical Fourier method for ensuring stability. This leads to a growth coefficient which can be made smaller than 1 in case of sufficiently small  $dt$ . This is a CFL like condition. Below we compose the computation by [16] to our system of equations. This enables use to state the following Theorem:

**Theorem 4.1** *Picard’s fixed point described in (4.4) contains a time step derived from the Sanz-Serna algorithm (4.2). If we want the growth function to be less than one, it is sufficient that the time step to be such that:*

$$dt < \frac{dx}{\pi(|\zeta^n|_{L^\infty} + |u^n|_{L^\infty})},$$

to ensure the stability of the Picard iterate.

**Proof :**

We can write Sanz-Serna’s Scheme (4.2) as:

$$A \begin{pmatrix} \hat{\zeta}_k^{n+1} \\ \hat{u}_k^{n+1} \end{pmatrix} = B \begin{pmatrix} \hat{\zeta}_k^n \\ \hat{u}_k^n \end{pmatrix} - \begin{pmatrix} \frac{ik dt}{4 + 2\epsilon dt |k|^\lambda} \mathcal{F}((\zeta^{n+1} + \zeta^n)(u^{n+1} + u^n)) \\ \frac{ik dt}{8(1 + k^2)} \mathcal{F}((u^{n+1} + u^n)^2) \end{pmatrix}, \quad (4.5)$$

where

$$A = \begin{pmatrix} 1 & \frac{ikdt}{2 + \epsilon dt |k|^\lambda} \\ \frac{ikdt}{2(1 + k^2)} & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{2 - \epsilon dt |k|^\lambda}{2 + \epsilon dt |k|^\lambda} & \frac{-ikdt}{2 + \epsilon dt |k|^\lambda} \\ \frac{-ikdt}{2(1 + k^2)} & 1 \end{pmatrix}. \quad (4.6)$$

Moreover, we have

$$\det(A) = 1 + \frac{k^2 dt^2}{4(1 + k^2)(1 + 0.5\epsilon dt |k|^\lambda)} > 0.$$

Then, the matrix  $A$  is invertible for all  $dt$ .

Now, we use the Picard iterate. We get

$$A \begin{pmatrix} \hat{\zeta}^{n+1} - \hat{v}_1^{m+1} \\ \hat{u}^{n+1} - \hat{v}_2^{m+1} \end{pmatrix} = -\frac{ik dt}{2} \begin{pmatrix} \frac{1}{2 + \epsilon dt |k|^\lambda} \mathcal{F}((\zeta^{n+1} + \zeta^n)(u^{n+1} + u^n) - (v_1^m + \zeta^n)(v_2^m + u^n)) \\ \frac{1}{4(1 + k^2)} \mathcal{F}((u^{n+1} + u^n)^2 - (v_2^m + u^n)^2) \end{pmatrix}. \quad (4.7)$$

A linearization, near  $(\hat{\zeta}^n, \hat{u}^n)$ , of the nonlinear term leads to the approximation

$$A \begin{pmatrix} \hat{\zeta}^{n+1} - \hat{v}_1^{m+1} \\ \hat{u}^{n+1} - \hat{v}_2^{m+1} \end{pmatrix} \approx -\frac{ik dt}{2} \begin{pmatrix} \frac{1}{2 + \epsilon dt |k|^\lambda} \hat{B}_1 \\ \frac{1}{4(1 + k^2)} \hat{B}_2 \end{pmatrix}, \quad (4.8)$$

where,

$$\hat{B}_1 = \mathcal{F}(u^n(\zeta^{n+1} - v_1^m) + \zeta^n(u^{n+1} - v_2^m)) \quad \text{and} \quad \hat{B}_2 = \mathcal{F}(u^n(u^{n+1} - v_2^m)). \quad (4.9)$$

Now, we multiply (4.8) by  $A^{-1}$ . We get

$$\begin{pmatrix} \hat{\zeta}^{n+1} - \hat{v}_1^{m+1} \\ \hat{u}^{n+1} - \hat{v}_2^{m+1} \end{pmatrix} \approx -\frac{ik \, dt}{2\det(A)} \begin{pmatrix} \frac{1}{2 + \epsilon dt |k|^\lambda} \hat{B}_1 - \frac{ik dt}{4(1+k^2)(2 + \epsilon dt |k|^\lambda)} \hat{B}_2 \\ \frac{-ik dt}{2(1+k^2)(2 + \epsilon dt |k|^\lambda)} \hat{B}_1 + \frac{1}{4(1+k^2)} \hat{B}_2 \end{pmatrix}. \quad (4.10)$$

By the Parseval's formula, we get

$$|\zeta^{n+1} - v_1^{m+1}|_{L^2} + |u^{n+1} - v_2^{m+1}|_{L^2} \leq \frac{|k| \, dt}{2} \left[ \frac{4(1+k^2) + 3|k|dt + 2 + \epsilon dt |k|^\lambda}{4(1+k^2)(2 + \epsilon dt |k|^\lambda)} \right] (|B_1|_{L^2} + |B_2|_{L^2}).$$

By (4.9), we have

$$|B_1|_{L^2} \leq |u^n|_{L^\infty} |\zeta^{n+1} - v_1^m|_{L^2} + |\zeta^n|_{L^\infty} |u^{n+1} - v_2^m|_{L^2} \quad \text{and} \quad |B_2|_{L^2} \leq |u^n|_{L^\infty} |u^{n+1} - v_2^m|_{L^2}.$$

Then, we get

$$|\zeta^{n+1} - v_1^{m+1}|_{L^2} + |u^{n+1} - v_2^{m+1}|_{L^2} \leq F(k) (|\zeta^{n+1} - v_1^m|_{L^2} + |u^{n+1} - v_2^m|_{L^2}),$$

where

$$\begin{aligned} F(k) &= \frac{|k| \, dt}{2} \left[ \frac{4(1+k^2) + 3|k|dt + 2 + \epsilon dt |k|^\lambda}{4(1+k^2)(2 + \epsilon dt |k|^\lambda)} \right] (|\zeta^n|_{L^\infty} + |u^n|_{L^\infty}) \\ &\leq |k| \, dt (|\zeta^n|_{L^\infty} + |u^n|_{L^\infty}), \end{aligned}$$

but we have  $|k| < \pi/dx$ , then

$$F(k) < \frac{\pi \, dt}{dx} (|\zeta^n|_{L^\infty} + |u^n|_{L^\infty}).$$

Therefore,  $F(k) < 1$  if

$$dt < \frac{dx}{\pi (|\zeta^n|_{L^\infty} + |u^n|_{L^\infty})},$$

and the proof is complete. □

### **Validating the program of this algorithm**

To simplify the notation, let  $\Upsilon$  be the nonlinear partial differential operator defined by the system (1.2), i.e.

$$(1.2) \quad \iff \quad \Upsilon(\zeta_\epsilon, u_\epsilon) = (0, 0). \quad (4.11)$$

To verify the algorithm, we study the system :

$$\Upsilon(\zeta_\epsilon, u_\epsilon) = \Upsilon(\zeta_{exact}, u_{exact}), \quad (4.12)$$

where

$$\zeta_{exact}(x, t) = \sin(t) \cos(2\pi x/L) \quad \text{and} \quad u_{exact}(x, t) = \cos(t) \sin(2\pi x/L).$$

We use these parameters:  $\epsilon = 0$ ,  $\lambda = 1$ ,  $N_x = 1024$  on the interval  $[-L/2, L/2]$  with  $L = 4\pi$  and  $T_{final} = 2$ . We compute the errors

$$ERR_{L^2} = \sqrt{|\zeta_\epsilon - \zeta_{exact}|_{L^2}^2 + |u_\epsilon - u_{exact}|_{L^2}^2} \quad \text{and} \quad ERR_{L^\infty} = \frac{1}{2}(|\zeta_\epsilon - \zeta_{exact}|_\infty + |u_\epsilon - u_{exact}|_\infty).$$

The results presented here have been obtained using Matlab software and the FFT matlab's function. We use the Sanz-Serna scheme and the standard Picard iterate which are defined in (4.2) and (4.4) to compute the solution of (4.12). Then we compute the errors.

The results can be seen in Table 1 and Figure 1.

$dt$	$ERR_{L^2}$	$ERR_{L^\infty}$
0.01	$2e - 5$	$6.07e - 6$
0.0075	$1.12e - 5$	$3.4e - 6$
0.005	$5e - 6$	$1.52e - 6$
0.0025	$1.24e - 6$	$3.78e - 7$
0.001	$2e - 7$	$6e - 8$
0.0005	$5e - 8$	$1.5e - 8$

Table 1

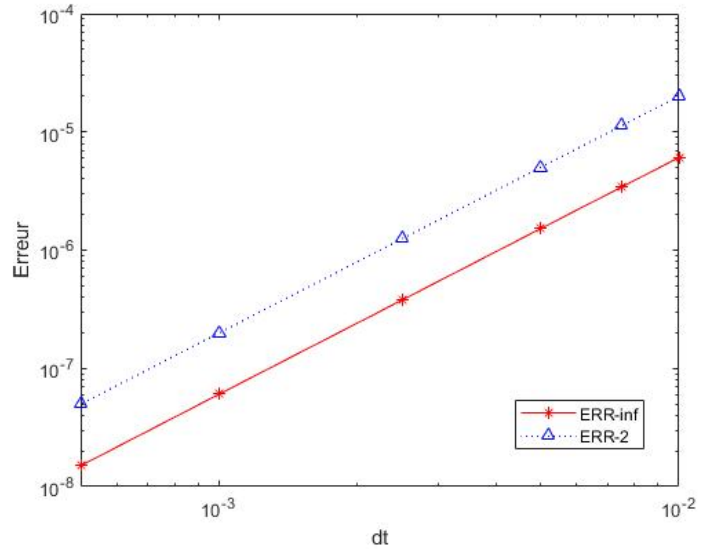


Figure 1: Variations of errors with respect to  $\epsilon$

We conclude that the solution  $(\zeta_\epsilon, u_\epsilon)$  of the system (1.2) tends to the exact solution  $(\zeta_{exact}, u_{exact})$  when  $dt$  tends to zero. So we can conclude that our numerical algorithm does what we claim it does.

## 4.2 Convergence of the regularized solution to the non-regularized one

We still use, below, the Sanz-Serna scheme and the standard Picard iterate which are defined in (4.2) and (4.4). Simulations are made for  $\lambda = 1$ , starting from the initial height  $\zeta_0 = 0$  and the initial velocity  $u_0 = \cos(2\pi x/L)$  on the interval  $[-L/2, L/2]$  with  $L = 4\pi$ , discretized in  $N_x = 1024$  points. The time step is  $dt = 5.10^{-5}$  until time  $T_{final} = 2$ .

To verify the convergence of the solution  $(\zeta_\epsilon, u_\epsilon)$  of the regularized Boussinesq system (1.2) to the solution  $(\zeta, u)$  of Boussinesq system (1.1). We compute

$$ERR_s = \sqrt{|\zeta_\epsilon - \zeta|_{H^s}^2 + |u_\epsilon - u|_{H^{s+1}}^2} \quad \text{and} \quad \delta_1^\epsilon = \frac{1}{2}(|\zeta_\epsilon - \zeta|_\infty + |u_\epsilon - u|_\infty),$$

with respect to  $\epsilon$ . We take here  $s = 1$ . We show the results in Table 2 and Figure 2 :

$\epsilon_i$	$\delta_1^\epsilon$	$ERR_s$
$10^{-1}$	$2.065e-2$	$1.2059$
$10^{-1.5}$	$6.75e-3$	$3.823e-1$
$10^{-2}$	$2.133e-3$	$1.21e-1$
$10^{-2.5}$	$6.899e-4$	$3.823e-2$
$10^{-2.75}$	$3.826e-4$	$2.147e-2$
$10^{-3}$	$2.152e-4$	$1.214e-2$

Table 2

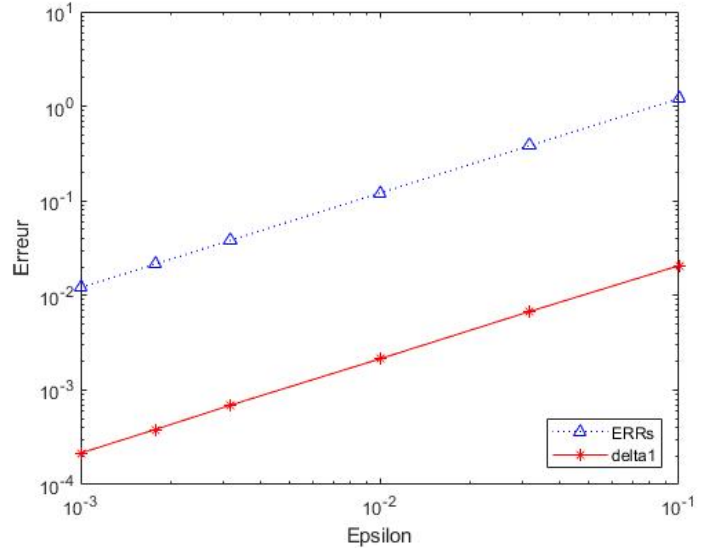


Figure 2: Variations of errors with respect to  $\epsilon$

In these results we see that the errors tend to zero when  $\epsilon$  tend to zero. So, we verify numerically the result of [9] that the solution of the regularized Boussinesq system (1.2) tends to the solution of the Boussinesq system (1.1) when  $\epsilon$  tends to zero.

### 4.3 Existence of an asymptotic expansion

In the theoretical part, we proved the existence of an asymptotic expansion of the solution of the regularized system (1.2) for  $m$  fixed. Now, we will verify this existence numerically for  $m = 2$ . We must calculate the rate of the convergence which is studied above as  $\epsilon$  tends to 0.

For  $a, b \in \mathbb{R}$ , we define the operator  $\Phi$  by

$$\Phi(a, b) = \sum_{i=1}^n (\log_{10}(ERR_1) - a \cdot \log_{10}(\epsilon_i) - b)^2. \quad (4.13)$$

To calculate the convergence rate, we minimise this operator. So we take

$$\begin{cases} \partial_a \Phi = 0 \\ \partial_b \Phi = 0 \end{cases} \implies \begin{cases} \sum_{i=1}^n \log_{10}(\epsilon_i) (\log_{10}(ERR_1) - a \cdot \log_{10}(\epsilon_i) - b) = 0, \\ \sum_{i=1}^n (\log_{10}(ERR_1) - a \cdot \log_{10}(\epsilon_i) - b) = 0. \end{cases}$$

We take the result of the Table 2, we get

$$\begin{cases} -30.0625a + 12.75b = -38.5847, \\ 12.75a - 6b = 16.772. \end{cases}$$

Then

$$a = 0.992 \quad \text{and} \quad b = -0.688.$$

We conclude that the solution  $(\zeta_\epsilon, u_\epsilon)$  tends at order 1 to the solution  $(\zeta, u)$  of the Boussinesq system (1.1) when  $\epsilon \rightarrow 0$ . So we numerically verified that:

$$(\zeta_\epsilon, u_\epsilon) = (\zeta, u) + o(1).$$

Then we can assume an asymptotic expansion up to order 1:

$$(\zeta_\epsilon, u_\epsilon) = (\zeta, u) + \epsilon(\zeta_1, u_1) + o(\epsilon).$$

and replace it in the system (1.2). This gives us the system satisfied by the coefficient  $(\zeta_1, u_1)$  (it is the same as the system (3.4) with  $m = 1$ ):

$$\begin{cases} (\zeta_1)_t + (u_1)_x + (\zeta \cdot u_1)_x + (\zeta_1 \cdot u)_x + g_\lambda[\zeta] = \frac{1}{\epsilon} o(\epsilon), \\ (u_1)_t - (u_1)_{txx} + (\zeta_1)_x + (u \cdot u_1)_x = \frac{1}{\epsilon} o(\epsilon). \end{cases} \quad (4.14)$$

To solve this system numerically, we drop the remaining term and we start from the initial height  $\zeta_1(0) = 0$  and the initial velocity  $u_1(0) = 0$ . We apply the Fourier transform in space and the Sanz-Serna scheme in time. We obtain:

$$\begin{cases} \widehat{\zeta}_{1k}^{n+1} = \widehat{\zeta}_{1k}^n - \frac{1}{2} ik dt \cdot (\widehat{u}_{1k}^{n+1} + \widehat{u}_{1k}^n) - \frac{1}{4} ik dt \cdot \mathcal{F}((\zeta^{n+1} + \zeta^n)(u_1^{n+1} + u_1^n)) \\ \quad - \frac{1}{4} ik dt \cdot \mathcal{F}((\zeta_1^{n+1} + \zeta_1^n)(u^{n+1} + u^n)) - \frac{1}{2} dt \cdot |k|^\lambda (\widehat{\zeta}_k^{n+1} + \widehat{\zeta}_k^n), \\ (1 + k^2) \widehat{u}_{1k}^{n+1} = (1 + k^2) \widehat{u}_{1k}^n - \frac{1}{2} ik dt \cdot (\widehat{\zeta}_{1k}^{n+1} + \widehat{\zeta}_{1k}^n) \\ \quad - \frac{1}{4} ik dt \cdot \mathcal{F}((u_1^{n+1} + u_1^n)(u^{n+1} + u^n)). \end{cases} \quad (4.15)$$

We use the Picard iterate (4.4). We compute the error  $\delta_2^\epsilon = \frac{1}{2}(|u_\epsilon - u - \epsilon \cdot u_1|_\infty + |\zeta_\epsilon - \zeta - \epsilon \cdot \zeta_1|_\infty)$  according to  $\epsilon$ . We get the results in Table 3 and Figure 3.

$\epsilon_i$	$\delta_2^\epsilon$
$10^{-1}$	$8.97e-4$
$10^{-1.5}$	$9.295e-5$
$10^{-2}$	$9.381e-6$
$10^{-2.5}$	$9.409e-7$
$10^{-2.75}$	$2.977e-7$
$10^{-3}$	$9.418e-8$

Table 3

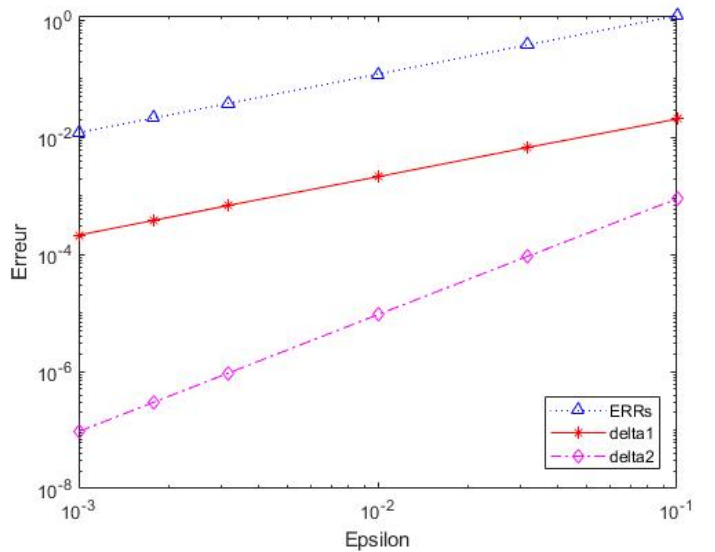


Figure 3: Variations of errors with respect to  $\epsilon$

Now, we use the operator (4.13) to compute the convergence rate of  $\delta_2^\epsilon$ . We use the results of the Table 3 and we get

$$a = 1.991 \quad \text{and} \quad b = -1.05.$$

So, the error  $\delta_2^\epsilon$  tends at order 2 to zero when  $\epsilon$  tends to 0. So we numerically verified the case  $m = 1$  in Theorem 3.1. Then we can assume an asymptotic expansion up to order 2:

$$(\zeta_\epsilon, u_\epsilon) = (\zeta, u) + \epsilon(\zeta_1, u_1) + \epsilon^2(\zeta_2, u_2) + o(\epsilon^2).$$

We replace this value of  $(\zeta_\epsilon, u_\epsilon)$  in the system (1.2). We obtain the system satisfied by the coefficient  $(\zeta_2, u_2)$  (it is the same as the system (3.4) with  $m = 2$ ):

$$\begin{cases} (\zeta_2)_t + (u_2)_x + (\zeta \cdot u_2)_x + (\zeta_1 \cdot u_1)_x + (\zeta_2 \cdot u)_x + g_\lambda[\zeta_1] = \frac{1}{\epsilon^2} o(\epsilon^2), \\ (u_2)_t - (u_2)_{txx} + (\zeta_2)_x + (u \cdot u_2)_x + u_1 \cdot (u_1)_x = \frac{1}{\epsilon^2} o(\epsilon^2). \end{cases} \quad (4.16)$$

We drop the remaining term and we start from the initial height  $\zeta_2(0) = 0$  and the initial velocity  $u_2(0) = 0$ . To solve this system numerically, we apply the Fourier transform in space and we use the Sanz-Serna scheme for the discretisation in time. We get

$$\begin{cases} \widehat{\zeta}_{2k}^{n+1} = \widehat{\zeta}_{2k}^n - \frac{1}{2} ik dt \cdot (\widehat{u}_{2k}^{n+1} + \widehat{u}_{2k}^n) - \frac{1}{4} ik dt \cdot \mathcal{F}((\zeta^{n+1} + \zeta^n)(u_2^{n+1} + u_2^n)) \\ \quad - \frac{1}{4} ik dt \cdot \mathcal{F}((\zeta_1^{n+1} + \zeta_1^n)(u_1^{n+1} + u_1^n)) \\ \quad - \frac{1}{4} ik dt \cdot \mathcal{F}((\zeta_2^{n+1} + \zeta_2^n)(u^{n+1} + u^n)) - \frac{1}{2} dt \cdot |k|^\lambda (\widehat{\zeta}_{1k}^{n+1} + \widehat{\zeta}_{1k}^n), \\ (1 + k^2) \widehat{u}_{2k}^{n+1} = (1 + k^2) \widehat{u}_{2k}^n - \frac{1}{2} ik dt \cdot (\widehat{\zeta}_{2k}^{n+1} + \widehat{\zeta}_{2k}^n) \\ \quad - \frac{1}{4} ik dt \cdot \mathcal{F}((u^{n+1} + u^n)(u_2^{n+1} + u_2^n)) - \frac{1}{8} ik dt \cdot \mathcal{F}((u_1^{n+1} + u_1^n)^2). \end{cases} \quad (4.17)$$

We compute the error

$$\delta_3^\epsilon = \frac{1}{2} (|u_\epsilon - u - \epsilon \cdot u_1 - \epsilon^2 \cdot u_2|_\infty + |\zeta_\epsilon - \zeta - \epsilon \cdot \zeta_1 - \epsilon^2 \cdot \zeta_2|_\infty),$$

according to  $\epsilon$ . So, by using the Picard iterate (4.4), we get the results in Table 4 and Figure 4:

$\epsilon_i$	$\delta_3^\epsilon$
$10^{-1}$	$3.925e-5$
$10^{-1.5}$	$1.278e-6$
$10^{-2}$	$4.0807e-8$
$10^{-2.5}$	$1.294e-9$
$10^{-2.75}$	$2.3e-10$
$10^{-3}$	$4.0827e-11$

Table 4

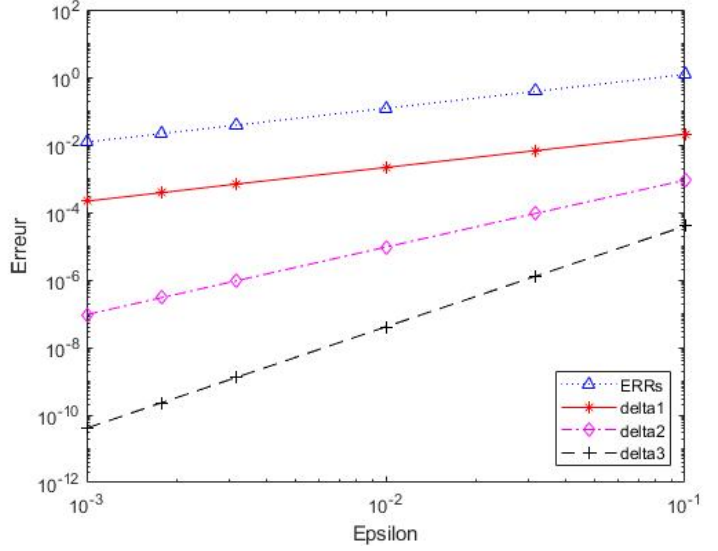


Figure 4: Variations of errors with respect to  $\epsilon$

We use the operator (4.13) to compute the convergence rate of  $\delta_3^\epsilon$ , and we use the results of the Table 4. We get

$$a = 2.992 \quad \text{and} \quad b = -1.409.$$

Then, the convergence rate of  $\delta_3^\epsilon$  is equal to 3 (see Figure 4).

Therefore, we justified numerically that the solution  $(\zeta_\epsilon, u_\epsilon)$  to the Cauchy problem associated with (1.2) has, in  $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ , an asymptotic expansion up to order 2,

$$(\zeta_\epsilon, u_\epsilon) = (\zeta, u) + \sum_{i=1}^2 \epsilon^i (\zeta_i, u_i) + o(\epsilon^2).$$

## 5 Conclusion

We have shown theoretically, in Theorem 3.1, that the solution  $(\zeta_\epsilon, u_\epsilon)$  to the regularized Boussinesq system (1.2) has an asymptotic expansion, at a given order  $m$ , in  $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$  with respect to the regularizing parameter  $\epsilon$ .

Then, by using the Sanz-Serna scheme, the standard Picard iterate and FFT procedure, we checked numerically the existence of this asymptotic expansion up to order 2.

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