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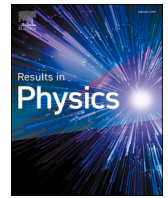
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Entanglement between Micro-Magnetism, electromagnetism and the Tensor Magnetic Phase Theory (TMPT) – Symmetry, conservation and invariance laws analysis at low frequency

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ABSTRACT

Ferromagnetic materials show magnetic structures with domains and walls. Thanks to decades of research regarding the origin and behaviour of magnetic domains, we now possess a general foundation which has been verified experimentally in single crystals and powders. The governing equations at the microscopic scale were built in the 1960 s when Brown published calculations of the magnetic moments distribution inside domain walls. This micromagnetic theory uses the so called LLG 'Landau-Lifshitz-Gilbert' equation and can include the damping effects. The LLG equation requires a coupling to the field derived from energy contributions: exchange, anisotropy, magnetostriction, stray-field ... and the anti-eddy field. At the macroscopic scale, such behaviours are lumped in a homogenized magnetization law for the electromagnetic field equations inside larger polycrystals. Therefore, the inhomogeneous magnetic material nature is always ignored. The Tensor Magnetic Phase Theory (TMPT) describes the magnetic structure and the magnetization thanks to a tensor variable at a mesoscopic scale. The material structuring is then explained thanks to an energy balance which will be discussed.

This paper presents the results of investigations on entangled relationships between the micromagnetic theory, the electromagnetic theory and what is called the Tensor Magnetic Phase Theory (TMPT), which statistically describes the magnetic structures of soft ferromagnetic materials. It examines a connection between the TMPT and the LLG by deriving the main energy terms. Then, the TMPT must stay compatible and coupled to the Maxwell equations at low frequencies with volume and surface connections. Additionally, this paper investigates the way to derive the domains structuring and magnetization laws through the Lagrange principle and the corresponding conservation laws with invariants linked to the Nøther theorem. Finally, the TMPT must be discussed while checking its coherence and formulation when changing the reference frame.

GLOSSARY (main symbols used in this paper)

Symbols	Descriptions
γ	gyromagnetic factor [$C.kg^{-1}$] ($\gamma = 28 \text{ THz} \cdot T^{-1}$ for the electron)
g	Landé factor [n.u.] ($g \approx 2$ for the electron)
f_L	Larmor frequency ($f_L = \gamma \cdot B / (2\pi)$)
μ_B	Bohr Magnetron [$A \cdot m^2$] ($\mu_B = 9.27 \cdot 10^{-24} A \cdot m^2$)
h	Planck constant [$J \cdot s$] ($h = 6.62 \cdot 10^{-34} J \cdot s$)
\hbar	$h / (2\pi)$
Z_c	number of atoms per lattice cell [n.u.]

(continued)

Symbols	Descriptions
$d\vec{s}$	spins of electrons [n.u.]
$d\vec{\mu}$	magnetic moment [$A \cdot m^2$]
\vec{m}	nanoscopic magnetization [$A \cdot m^{-1}$]
\vec{h}_{eff}	local efficient magnetic field [$A \cdot m^{-1}$]
$\vec{h}_{eff} = \vec{h}_{ex} + \vec{h}_{an} + \vec{h}_s + \vec{h} + \vec{h}_{dm} + \vec{h}_{ed} + \vec{h}$	(ex: magnetic exchange, an: magneto-crystalline anisotropy, λs : stress induced anisotropy, λe : self magneto-striction, dm: stray field demagnetization, ed: eddy currents) field [$A \cdot m^{-1}$]
\vec{h} and \vec{H}	magnetic excitation field at the nano and macroscale [$A \cdot m^{-1}$]

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(continued)

Symbols	Descriptions
\vec{e} and \vec{E}	electrical excitation field at the nano and macroscale [$V.m^{-1}$]
\vec{b} and \vec{B}	magnetic induction or flux density at the nano and macroscale [T]
\vec{m} and \vec{M}	magnetization at the nano and macroscale [$A.m^{-1}$]
$\vec{\gamma}$ and \vec{J}	magnetic polarization at the nano and macroscale [T]
$[\Lambda^2]$	tensor magnetic phase variable [m^2]
$\vec{\varphi}$	elementary magnetic flux vector associated to [Λ^2] [Wb]
ϵ_0	dielectric permittivity of vacuum or air [$F.m^{-1}$] ($\epsilon_0 = 8.85.10^{-12} F.m^{-1}$)
μ_0	magnetic permeability of vacuum or air [$H.m^{-1}$] ($\mu_0 = 4\pi.10^{-7} H.m^{-1}$)
σ	local electrical conductivity [$S.m^{-1}$]
$[\nu]$	local magnetic reluctivity at the macroscopic scale [$m.H^{-1}$]
τ_b	macroscopic time delay of the magnetic flux density B [s]
M_s	saturation magnetization [$A.m^{-1}$]
J_s	saturation magnetic polarization [T] ($J_s = \mu_0 M_s$)
A_{ex}	magnetic exchange constant at the nanoscopic scale [$J.m.A^{-2}$]
γ_w	integral of nanoscopic walls energy, exchange and anisotropy [$J.m^{-2}$]
K_{an}	Magneto-crystalline anisotropy constant at the nanoscopic scale [$J.m^{-3}$]
\vec{K}_{an}	macroscopic magnetic anisotropy vector [$J.m^{-3}$], K_{an} anisotropy constant at the macroscopic scale [$J.m^{-3}$]

(continued)

Symbols	Descriptions
$[\lambda]$	tensor of magnetostriction coefficients (see appendix 4) [n.u.]
$[s]$	external stress tensor [Pa]
λ_{100}	magnetostriction coefficient of a cubic crystal along [100] [n.u.]
λ_{111}	magnetostriction coefficient of a cubic crystal along [111] [n.u.]
C_{ex}	magnetic exchange coefficient for the TMPT [n.u.]
C_{an}	magnetic anisotropy coefficient for the TMPT [n.u.]
C_{is}	stress induced magneto-strictive anisotropy coefficient for the TMPT [n.u.]
$C_{\delta\epsilon}$	self magneto-striction coefficient for the TMPT [n.u.]
C_{dm}	demagnetizing coefficient [n.u.]
τ	mesoscopic time delay of the magnetic structuring into domains [s]
$\ell, \vec{\mathcal{L}}$ and L	Lagrangian energy densities for the micro-magnetism, TMPT and electromagnetism [$J.m^{-3}$]
$t, \vec{\mathcal{T}}$ and T	kinetic energy densities for the micro-magnetism, TMPT and electromagnetism [$J.m^{-3}$]
$u, \vec{\mathcal{U}}$ and U	potential energy densities for the micro-magnetism, TMPT and electromagnetism [$J.m^{-3}$]

NOTATIONS
 A symbol ξ with thin letters denotes the scalar ξ , $\vec{\xi}$ denotes a 3D vector and $[\xi]$ denotes a 3D tensor.

Introduction

Ferromagnetic materials show magnetic structures with domains and walls [1–7]. Thanks to decades of research regarding the origin and behaviour of magnetic domains, we now possess a general foundation which has been verified experimentally in single crystals and powders [8–15]. The governing equations at the microscopic scale were built in

the 1960s when Brown published calculations of the magnetic moments’ distribution inside domain walls [19–22]. This micromagnetic theory uses the so called “Landau-Lifshitz-Gilbert” equation (LLG) and can include the damping effects [23]. The LLG equation requires a coupling to the field derived from energy contributions: exchange, anisotropy, magnetostriction, stray-field ... and the anti-eddy field. At the macroscopic scale, such behaviours are lumped in a homogenized

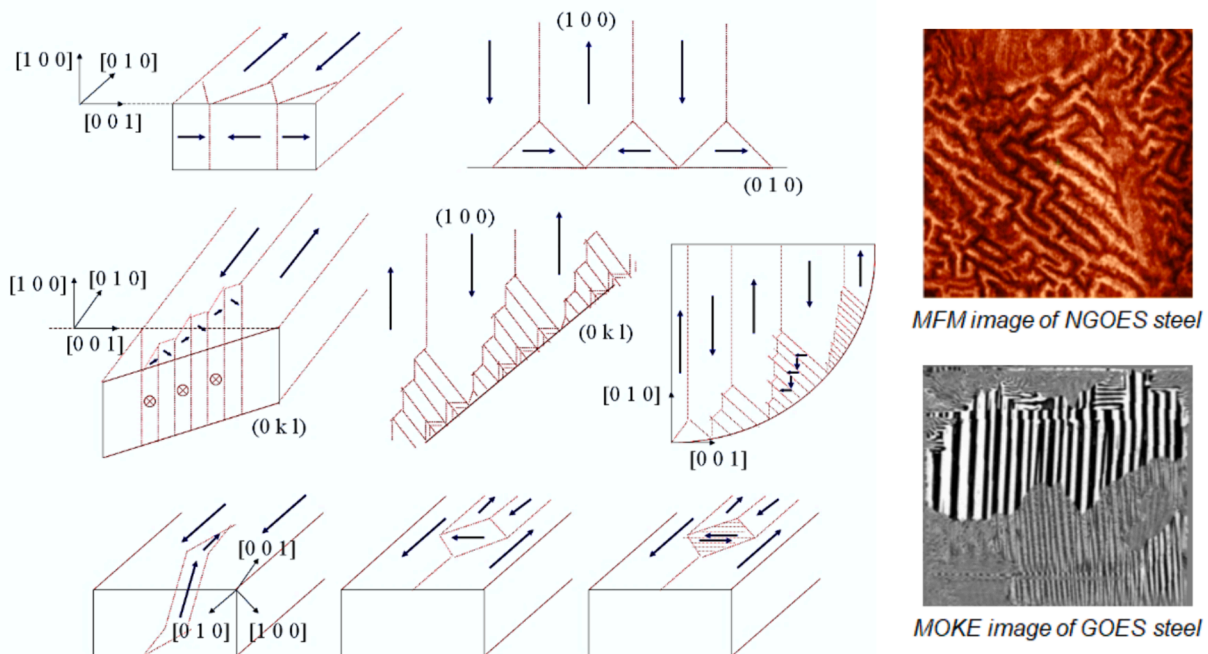


Fig. 1. representation of usual shapes and orientations of magnetic domains in the magnetic structure of classical electrical steels made of iron and silicon SiFe (GOES and NGOES) from references [3,28].

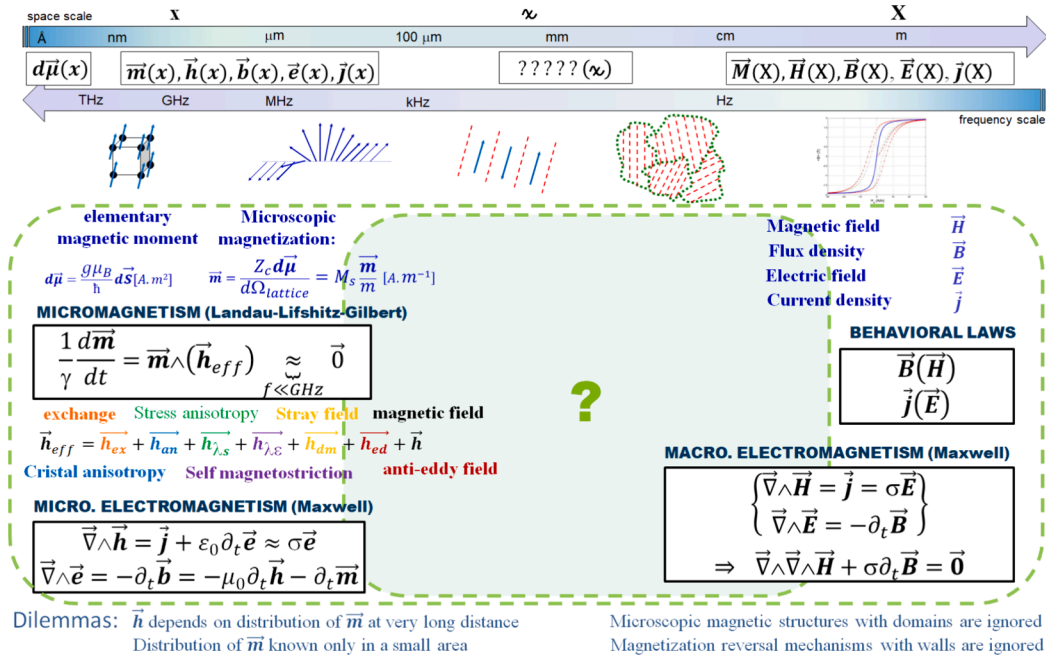


Fig. 2. State of the Art on the description of electromagnetic properties at different space and frequency scales inside soft magnetic materials with magnetic moments and a domains' structure.

magnetization law for the electromagnetic field equations inside larger polycrystals [16–18,24–27]. Therefore, the inhomogeneous magnetic material nature is always ignored. The Tensor Magnetic Phase Theory (TMPT) describes the magnetic structure and the magnetization thanks to a tensor variable at a mesoscopic scale [28]. The material structuring is then explained owing to an energy balance which will be discussed. This work examines a connection between the TMPT and the LLG by deriving the main energy terms. Then, the TMPT must stay compatible and coupled to the Maxwell equations with volume and surface connections [28–30,40]. Additionally, this paper investigates the way to derive the domains structuring and magnetization laws through the Lagrange principle and the corresponding conservation laws with invariants linked to the Noether theorem [31,32]. Finally, the TMPT must be discussed while checking its coherence and formulation when changing the reference frame [33–39]. The main reason for this research is the heterogeneity of polarized materials with subdivisions called domains separated by walls emphasized by several physicists in the 20th century [1–7]. Investigations of the present work have been initiated in [28] by looking first at the usual soft magnetic materials such as Non Grain Oriented Electrical Steels (NGOES) and Grain Oriented Electrical Steels (GOES). The Fig. 1 gives a schematic representation of several kinds of magnetic structures including usual shapes of polarized domains inspired from true images that can be found in the literature [3]. Additionally, this kind of considerations may also be applied to other polarizations in ferroelectric and multiferroic materials (see § 6.2).

In the following, the context, brief state of the art and dilemma are introduced. The core of the work is dedicated first to the introduction, justification and description of the so called Tensor Magnetic Phase Theory (or in brief TMPT), then to its relationship or coupling with the classical electromagnetism theory, until the derivation of main governing equations. Before giving some realistic test cases and a short discussion on the results, some symmetry and invariance properties are looked at.

From local and deterministic to “Holistic” paradigm

This work examines a connection between the TMPT and the LLG by deriving the main energy terms.

State of the art and dilemma

The literature in the field is totally informative at either very small dimensions (typically the nanoscale x) or quite larger dimensions (typically the macroscale X) (see Fig. 2). The physical description and the corresponding mathematical tools used to study those kinds of materials depend on the observation scale. Let's examine the phenomenon at the different space scales, to be often associated to a time or frequency scale (see Fig. 2).

At atomic scale, the origin and mutual interactions of elementary magnetic moments are described through a group of atoms in a lattice cell ($d\Omega_{lattice}$) or a molecule pattern. The properties such as the saturation magnetization, the exchange constant between moments or the anisotropy constant are defined [4–6]. The magnetic moments are carried by electrons (free or bound, following the models given in [4–7]) and are due to orbital moments or spins and to their interactions (spin–orbit coupling [7]).

At the nanoscopic scale, matter can be viewed as elementary volumes with nanoscopic dimensions, containing several magnetic moments, that can adopt many different configurations [8–10]. To simplify, the spatial distribution of moments can be either uniform (domains) or rotational (walls, vortex etc.).

At the microscopic scale, we start observing at distance the characteristic magnetic structure of a material. The latter depends on several microscopic physical properties (metallography, anisotropy, exchange etc.) and on the local geometrical discontinuities (interfaces, boundaries etc.). Magnetic structures are arrangements of domains and walls or other magnetic objects (different types of domains and vortices) [11–15]. This comes from a compromise found which minimizes the sum of micromagnetic energies including the magneto-crystalline anisotropy, magnetic exchange, demagnetizing stray field energy and sometimes the magnetostriction.

At the mesoscopic scale, statistical averaging of several microscopic domains and walls is possible as long as the parts contain enough metallographic grains. At this scale, statistical tools such as the magnetic phase theory [8,16] can be used. The small details inside the magnetic structure are neglected and lumped in mean behavioural laws, loss models [17,18], included in the electromagnetic field equations [27].

The last observation scale is the macroscopic scale, the one of the considered entire part or specimen, for which the macroscopic behaviour, with the hysteresis loops and losses are observed and measured [17].

At a deeper level, at a nanoscale, micro-magnetism and electro-magnetism are usually used. Micro-magnetic problems are governed by the Landau-Lifshitz-Gilbert equation (LLG) [19–22] associated to the Maxwell equations with electromagnetic fields defined at the nanoscopic or/and microscopic scale(s). Several authors at the beginning of the 20th century introduced and developed the Landau-Lifshitz-Gilbert equation with conservative and dissipative contributions [23]. It describes the quasi-static and dynamical behaviour of a single spin $\vec{d}s$, magnetic moment $\vec{d}\mu = (g\mu_B/\hbar)\vec{d}s$ or microscopic magnetization vector $\vec{m} = (Z_c/d\Omega_{\text{lattice}})\vec{d}\mu$ ($|\vec{m}| = M_s$) within a local effective magnetic field \vec{h}_{eff} (1) generated by all neighbours, the environment and the magnetic field source \vec{h} . The latter total effective magnetic field is usually written as the sum of several magnetic contributions (the quasi-static contributions giving $\vec{h}_{\text{qs,eff}}$: magnetic exchange between atoms \vec{h}_{ex} , magneto-crystalline anisotropy \vec{h}_{an} , stress induced magnetic anisotropy \vec{h}_{is} , self magnetostriiction \vec{h}_{is} and stray demagnetizing field \vec{h}_{dm} plus the dynamic contribution, i.e. the anti-eddy field \vec{h}_{ed}) that are at the origin of the micro-magnetism [19–23]:

$$\vec{h}_{\text{eff}} = \vec{h} + \vec{h}_{\text{ex}} + \vec{h}_{\text{an}} + \vec{h}_{\text{is}} + \vec{h}_{\text{dm}} + \vec{h}_{\text{ed}} = \vec{h}_{\text{qs,eff}} + \vec{h}_{\text{ed}} \quad (1a)$$

$$\vec{h}_{\text{ed}} = -\frac{\alpha}{\gamma M_s} \frac{d\vec{m}}{dt} = -\eta \frac{d\vec{m}}{dt} \quad (1b)$$

The time delay constant $\eta = \alpha\gamma^{-1}M_s^{-1}$ quantifies the average anti-eddy field damping throughout the sample [23]. Then the (LLG) Eq. (2) tells us that the effective field exerts a magnetic torque:

$$\frac{1}{\gamma} \frac{d\vec{m}}{dt} = \vec{m} \wedge \left(\vec{h}_{\text{eff}} \right) = \vec{m} \wedge \left(\vec{h}_{\text{qs,eff}} \right) - \vec{m} \wedge \left(\frac{\alpha}{\gamma M_s} \frac{d\vec{m}}{dt} \right) \xrightarrow{f \ll f_L} \vec{0} \quad (2)$$

The first term with time derivation is responsible for the Larmor precession [25,26] at a frequency called the Larmor frequency given by $f_L = \gamma B/(2\pi)$ with $\gamma = 28.10^{12} \gg 1$. As long as we focus on time varying fields with low frequencies ($f/f_L \ll 1$), this first term tends towards zero and can be neglected. This Eq. (2) is at the basis of the micromagnetic behaviour within magnetic domains [19] and walls [22].

Part of the dilemma at the nano or microscale, is that the local magnetic field \vec{h}_{eff} , is needed and depends on the space and time distribution of \vec{m} at very long distance (Maxwell) (probably longer than the nanoscale at purpose). As a consequence, the micromagnetic problem cannot be solved without implementing the resolution of the microscopic Maxwell equation to find the electromagnetic fields within the complex and changeable magnetic structure. Unfortunately, the distri-

with low frequencies ($f \ll GHz$), homogenized material's behavioural laws and electromagnetic fields in matter are usually used through the Maxwell equations without wave propagation [24]. Then the microscopic magnetic structure with domains and walls is ignored and undescribed. Therefore, this paper aims to question Physics and demands a congruent solution to this problem.

Introduction of tensor magnetic phases at a mesoscopic scale

To make sure we ask the question correctly and that we will have a useful answer; let us introduce a phenomenological and ‘‘Holistic’’ way [48] to define the magnetization in such a polarized material at a mesoscopic scale (ℓ) (quite larger than the crystal size $\gg nm$), between the nanoscopic and macroscopic scales [28]. To do so, we apply a volume integral averaging procedure $\langle \cdot \rangle$ (3, 4), like in references [27,29,30], to express the magnetization $\vec{M} = \langle \vec{m} \rangle$ or polarization $\vec{J} = \mu_0 \vec{M} = (J_1, J_2, J_3)^T = \langle \vec{\gamma} \rangle$ that results from couples of domains α with total volume Ω_α ($\Omega_\alpha = \Omega_{\alpha\uparrow} + \Omega_{\alpha\downarrow}$), magnetically saturated in different and contradictory directions $\uparrow\downarrow$ ($\vec{m}_{\alpha\uparrow} = +M_s \vec{u}_{\alpha\uparrow}$ and $\vec{m}_{\alpha\downarrow} = -M_s \vec{u}_{\alpha\downarrow}$ or $\vec{\gamma}_{\alpha\uparrow\downarrow} = \mu_0 \vec{m}_{\alpha\uparrow\downarrow} = \pm J_s \vec{u}_{\alpha\uparrow\downarrow}$) and with two different volumes $\Omega_{\alpha\uparrow}$ and $\Omega_{\alpha\downarrow}$, and effective cross sections in the plane $(\vec{u}_{\alpha 2}, \vec{u}_{\alpha 3})$ perpendicular to the domains α magnetization direction $\pm \vec{u}_{\alpha 1}$.

$$\sum_\alpha \vec{J}_\alpha \left(\iiint_{\Omega_\alpha} d^3x \right) = \sum_\alpha \left(\iiint_{\Omega_{\alpha\uparrow}} \vec{\gamma}_{\alpha\uparrow} d^3x - \iiint_{\Omega_{\alpha\downarrow}} \vec{\gamma}_{\alpha\downarrow} d^3x \right) \quad (3)$$

Let's use the transfer matrix \mathbb{T}_{ux} from the reference frame $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ to the domains frame $(\vec{u}_{\alpha 1}, \vec{u}_{\alpha 2}, \vec{u}_{\alpha 3})$ such that $(\mathbb{T}_{ux})_{ij} = \vec{u}_{\alpha i} \cdot \vec{x}_j$; and the Jacobian matrix $[\mathbb{J}_{xu_\alpha}]$ such that $([\mathbb{J}_{xu_\alpha}])_{ij} = \partial_{u_{\alpha j}} x_i$.

$$\begin{aligned} & \left([\mathbb{T}_{u_\alpha x}] \iiint_{\Omega_\alpha} |\mathbb{J}_{xu_\alpha}| d^3u_\alpha \right) \cdot \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \left(\iiint_{\Omega_{\alpha\uparrow}} [\mathbb{T}_{u_\alpha x}] |\mathbb{J}_{xu_\alpha}| d^3u_\alpha \right) \\ & - \left(\iiint_{\Omega_{\alpha\downarrow}} [\mathbb{T}_{u_\alpha x}] |\mathbb{J}_{xu_\alpha}| d^3u_\alpha \right) \cdot \begin{pmatrix} J_s \\ J_s \\ J_s \end{pmatrix} \end{aligned} \quad (4)$$

We propose to introduce an original tensor variable called $[\Lambda^2]$ which is homogeneous to a squared length to be interpreted as the effective cross section of couples of domains relatively to one direction and depending on its polarization direction (5). The matrix inverse of this tensor will be called $[\mathbb{V}^2] = [\Lambda^2]^{-1}$ and when divided by the effective length of domains along its magnetization direction, its components can be interpreted as the volume densities of walls in the main directions of space [28].

$$[\Lambda^2] = \frac{\iiint_{\Omega_\alpha} |\mathbb{J}_{xu_\alpha}| d^3u_\alpha [\mathbb{T}_{u_\alpha x}]}{\int_{\Omega_\alpha} du_{\alpha 1}} = \frac{(\iiint_{\Omega_{\alpha\uparrow}} |\mathbb{J}_{xu_\alpha}| d^3u_\alpha + \iiint_{\Omega_{\alpha\downarrow}} |\mathbb{J}_{xu_\alpha}| d^3u_\alpha) [\mathbb{T}_{u_\alpha x}]}{\int_{\Omega_\alpha} du_{\alpha 1}} = [\Lambda_\uparrow^2] + [\Lambda_\downarrow^2] \quad (5)$$

bution of \vec{m} can be known only in small areas, insufficient to render the interactions at long distance and that is why, the micromagnetic problems with a solution are limited to very small nano-objects, like a wall or a vortex, or nano-crystals. At the intermediate macroscopic scales and

An elementary flux vector $\vec{\varphi}$ is also introduced (6) and depends on the magnetic saturation M_s or J_s and the difference of effective cross sections for the two opposite polarizations of neighbouring domains.

$$\vec{\varphi} = \frac{\left(\iint_{\Omega_{\alpha 1}} |\downarrow_{x u_{\alpha}}| d^3 u_{\alpha} - \iint_{\Omega_{\alpha 1}} |\downarrow_{x u_{\alpha}}| d^3 u_{\alpha} \right) [\mathbb{T}_{u_{\alpha} x}] \bullet J_s \bullet \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\int_{\Omega_{\alpha}} d u_{\alpha 1}} = ([\Lambda_{\uparrow}^2] - [\Lambda_{\downarrow}^2]) \bullet J_s \bullet \vec{1} \quad (6)$$

This holistic way considers the domains shapes and sizes with $\vec{J} = \mu_0 \vec{M}$ the average of $\vec{\gamma} = \mu_0 \vec{m}$ (see appendix 3), called the magnetic polarisation (7), which equals the multiplication of the tensor $[V^2]$, able to render the different magnetic phases for the domains with the elementary magnetic flux vector $\vec{\varphi}$.

$$\vec{J}(x, t) = \mu_0 \vec{M}(x, t) = \mu_0 \langle \vec{m}(x, t) \rangle_x = [V^2(x, t)] \vec{\varphi}(x, t) \quad (7)$$

Partial differential properties of $[V^2]$ and $\vec{\varphi}$ are as follow:

$$V_{ij}^2 = \frac{\partial J_i}{\partial \varphi_j} = \mu_0 \frac{\partial M_i}{\partial \varphi_j} \text{ and } \varphi_j = \frac{\partial J_i}{\partial V_{ij}^2} = \mu_0 \frac{\partial M_i}{\partial V_{ij}^2} \quad (8)$$

There is a connection with the magnetic phase volumes introduced by Louis Néel, Landau & Lifschitz or Kittel during the 20th century [11,18,21], but with neither any energy formulation nor clear state variable proposed yet. Now we propose the question: What would be the energy contributions and formulation leading to the governing equations of such a magnetic structure represented by $[V^2]$ or $[\Lambda^2]$ and $\vec{\varphi}$?

Lagrangian energy densities

To do so, let's build each energy contribution with the help of what already exists at the nanoscale. The micromagnetic theory represented by the Landau-Lifschitz-Gilbert equation (LLG), according to Gilbert

$$\mathcal{U}_{ex} = \langle u_{ex} \rangle = \langle -A_{ex}(\vec{m} \bullet \Delta(\vec{m})) \rangle \vec{\nabla} \bullet \vec{m} \xrightarrow{\rightarrow 0} \langle A_{ex} | (\vec{\nabla} \wedge \vec{m})|^2 \rangle \Rightarrow \vec{\mathcal{U}}_{ex} \vec{\nabla} \bullet \vec{J} \xrightarrow{\rightarrow 0} C_{ex} \frac{\gamma_w^6}{K_{an}^5} (|\vec{\nabla}| \times [V^2])^{\vec{\nabla}^2} \quad (15)$$

himself, can be derived from a Lagrangian \mathcal{L} [10,23,31] limited to minus the internal potential energy u when the frequency is low (quite lower than 1 GHz), which provides each field contribution ($\vec{h} = -\vec{\nabla}_{\vec{m}}(u)$). Each potential energy at the nanoscale is perfectly known as a function of \vec{m} , involving material properties such as the exchange and anisotropy constants and the saturation polarization.

$$\mathcal{L} = \mathcal{L} - u \approx -u (f \ll \text{GHz}) \quad (9)$$

$$u = u_{ex} + u_{an} + u_{is} + u_{\lambda e} + u_{dm} \quad (10)$$

Considering the magnetic structure with domains, it is possible to express equivalent energy contributions at a mesoscopic scale (11-13), involving the new tensor variable $[V^2]$ (inverse of $[\Lambda^2]$) and the flux vector $\vec{\varphi}$ through the magnetization big \vec{M} rather than \vec{m} .

$$\vec{\mathcal{L}} = \vec{\mathcal{F}} - \vec{\mathcal{U}} \approx -\vec{\mathcal{U}} (f \ll \text{GHz}) \quad (11)$$

$$\vec{\mathcal{U}} = \vec{\mathcal{U}}_{ex} + \vec{\mathcal{U}}_{an} + \vec{\mathcal{U}}_{is} + \vec{\mathcal{U}}_{\lambda e} + \vec{\mathcal{U}}_{dm} \quad (12)$$

We propose then to separate the problem of finding $[V^2]$ from the problem of finding $\vec{\varphi}$ by defining a vector energy density $\vec{\mathcal{F}}$ rather than a scalar energy density \mathcal{L} , for which each component is obtained by fixing the flux direction in the direction of this component only (14)

(each energy component i is proportional to the derivative of scalar energy as a function of the squared $\vec{\varphi}$ component i).

$$\mathcal{U}(x, t) = \langle u(x, t) \rangle_x \quad (13)$$

$$\mathcal{U}_i = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \mathcal{U} \quad (14)$$

To do so, we need the microscopic magnetic walls energy density $\gamma_w [J.m^{-2}]$ and an anisotropy vector $\vec{K}_{an} = K_{an} \vec{u}_{an} [J.m^{-3}]$ (uni-axial case), which re-establish the homogeneity. The factor $J_s^2 (\gamma_w / K_{an})^4$ contains both the physical dimensions and the microscopic information related to the microscopic exchange plus anisotropy energies inside the walls not included elsewhere in the TMPT. Each vector potential energy, only defined in [28], can then be derived with calculations detailed in the appendix 3.

Magnetic exchange energy

The first energy contribution is the magnetic exchange one (15), proportional to some extent to the squared magnetization rotational for the nanoscale, and therefore proportional to the squared $[V^2]$ rotational for the TMPT (see appendix 2-3 and reference [28]) (notice that the rotational of a tensor here remains a tensor, and the product used here between two tensors gives a vector, see appendix 1)

The origin and equivalent formulations of the magnetic exchange can be found in the appendix 2.

Magneto-Crystalline anisotropy

The second energy contribution is the magneto-crystalline anisotropy (16), proportional for uni-axial cases to the squared sinus of angle between \vec{m} and the anisotropy axis \vec{u}_{an} . This is translated for the TMPT with the squared difference between $[V^2]$ and its contraction along the anisotropy vector \vec{K}_{an} (appendix 3).

$$\begin{aligned} \mathcal{U}_{an} = \langle u_{an} \rangle &= \left\langle \frac{K_{an}}{M_s^2} \left| \left(\vec{m} \wedge \vec{u}_{an} \right) \wedge \vec{u}_{an} \right|^2 \right\rangle \Rightarrow \vec{\mathcal{U}}_{an} \\ &= C_{an} \frac{\gamma_w^4}{K_{an}^3} \left([V^2] - \left(\frac{\vec{K}_{an} \cdot ([V^2] \cdot \vec{K}_{an})^T}{|\vec{K}_{an}|^2} \right)^T \right)^{\vec{\nabla}^2} \end{aligned} \quad (16)$$

Self Magneto-Striction energy

Similar techniques can lead to an equivalent expression of the self magnetostriction energy ...involving the magnetostriction tensor $[\lambda^2]$ and the stiffness tensor $[C]$ (see appendix 3 and appendix 4).

$$\begin{aligned}\mathcal{U}_{\lambda\varepsilon} = \langle u_{\lambda\varepsilon} \rangle &= \left\langle +\frac{9}{4} \frac{[\vec{m} \otimes \vec{m}]}{M_s^2} \circ \left([\lambda^2] \cdot : [C] \cdot \frac{[\vec{m} \otimes \vec{m}]}{M_s^2} \right) \right\rangle \\ \Rightarrow \vec{\mathcal{U}}_{\lambda\varepsilon} &= +\frac{9}{4} C_{\lambda\varepsilon} \frac{\gamma_w^4}{K_{an}^3} [V^2] \odot \left(\frac{[\lambda^2] \cdot : [C]}{K_{an}} \cdot : [V^2] \right)\end{aligned}\quad (17)$$

Stress induced magneto-striction anisotropy

Similar techniques can lead to an equivalent expression of the stress induced magneto-striction anisotropy energy (17), involving the magnetostriction tensor $[\lambda]$ and the stress tensor $[s]$ (appendix 4).

$$\begin{aligned}\mathcal{U}_{\lambda s} = \langle u_{\lambda s} \rangle &= \left\langle -\frac{3}{2} \frac{[\vec{m} \otimes \vec{m}]}{M_s^2} \circ \left([\lambda] \cdot : [s] \cdot \frac{[\vec{m} \otimes \vec{m}]}{M_s^2} \right) \right\rangle \\ \Rightarrow \vec{\mathcal{U}}_{\lambda s} &= -\frac{3}{2} C_{\lambda s} \frac{\gamma_w^4}{K_{an}^3} [V^2] \odot \left(\frac{[\lambda] \cdot : [s]}{K_{an}} \cdot : [V^2] \right)\end{aligned}\quad (18)$$

The operator $\cdot :$ is a kind of tensor product between two tensors which yields a tensor (see Appendix 1).

Demagnetizing stray-field energy

The equivalent demagnetizing energy is minimized at the TMPT level by superimposing a non-divergence condition on $[\Lambda^2]$ and $[V^2]$ tensors (19–20);, from (5) (transfer matrices properties):

$$\vec{\nabla} \cdot [V^2] = \vec{0} \quad (19)$$

$$\vec{\nabla} \cdot [\Lambda^2] = \vec{0} \quad (20)$$

This non-divergence condition is closely related to an absence of volume equivalent magnetic charges given by $\vec{\nabla} \cdot \vec{J} \rightarrow 0$ inside the volume of the material. However, any infringement of this principle due to a local magnetization vector and tensor divergence ($\vec{\nabla} \cdot \vec{J} \neq 0$ and $\vec{\nabla} \cdot [V^2] \neq \vec{0}$) may generate a demagnetizing energy contribution $\propto -[V^2] \odot [\vec{\nabla}] (\vec{\nabla} \cdot [V^2])$. This term shall be balanced and compensated by an opposite magnetic exchange contribution ($\propto [V^2] \odot [\vec{\nabla}] (\vec{\nabla} \cdot [V^2])$) still in order to minimize the total energy. As a consequence and as expected, the demagnetizing effects depend mainly on the magnetic poles at the surface boundaries through a surface energy term (21) involving the domains polarization components perpendicular to the surface with normal direction \vec{n} .

$$\vec{\mathcal{U}}_{dm} \propto C_{dm} \frac{K_{an}^4}{\gamma_w^3} \left(\vec{n} \cdot ([\Lambda^2] \cdot \vec{n})^T \right)^{\odot 2} \quad (21)$$

Eddy currents' lost energy

Finally, we also have to consider a dissipation energy functional (22) representing the energy losses due to eddy currents, proportional to the

Entanglement between magnetic phases and electromagnetic fields

Then, the TMPT must stay compatible to the Maxwell equations with volume and surface connections.

Coupling between the TMPT and the Maxwell equations at borders

This tensor representation of a magnetic structure is perfectly compatible with the classical electromagnetic fields and corresponding Maxwell equations [24]. Independent of the medium, the flux density \vec{B} (23) can still be written as a function of the polarization \vec{J} and the magnetic field \vec{H} . \vec{J} can be written with $[V^2]$ and $\vec{\varphi}$ (7), considering $[V^2] = [0]$ for non ferromagnetic materials. The magnetic field \vec{H} (24) can still be considered as dependent on \vec{B} through a reluctivity $[\nu]$.

$$\vec{B} = \mu_0 \vec{H} + \vec{J} \quad (23)$$

$$\vec{H} = [\nu] \cdot (\vec{B} + \tau_b \partial_t \vec{B}) \quad (24)$$

The field diffusion can still be computed in air (25) and in the material (26), providing that $[V^2]$ is known.

$$\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{H} = \vec{0} \quad (25)$$

$$\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{H} + \sigma \partial_t \vec{B} = \vec{0} \quad (26)$$

$[V^2]$ can be computed by the TMPT providing a coupling formulation (27) with the fields at boundaries.

$$[\text{constraint}(\vec{H}, \vec{B}, [V^2])] = [0] \quad (27)$$

The Poynting theorem and flux diffusion principle

Looking at the Hamiltonian of electromagnetic field (or the Poynting theorem [24]),

$$\mathcal{H} = T + U = \int \left(\sigma^{-1} |\vec{\nabla} \wedge \vec{H}|^2 + \vec{H} \cdot \partial_t \vec{B} \right) . dt \quad (28)$$

we can notice at the first-order regarding the dynamic contributions ($|\tau_b \partial_t \vec{B}| \ll |\vec{B}|$), that some of the TMPT energy contributions might be closely related to the part of Hamiltonian that comes from the space derivative of the reluctivity (for anisotropic properties (29)) and the tensor $[V^2]$ (for the exchange properties (29)) or its time derivative (for the dissipation functional (30))

$$\begin{aligned}\vec{\nabla} \wedge \vec{H} \approx \vec{\nabla} \wedge ([\nu][V^2]\vec{\varphi}) &= \underbrace{\vec{\nabla} \wedge ([\nu])[V^2]\vec{\varphi}} + \underbrace{[\nu]\vec{\nabla} \wedge ([V^2])\vec{\varphi}} + [\nu][V^2]\vec{\nabla} \wedge (\vec{\varphi}) \\ &= \vec{U}_{an} \frac{J_{\sigma}^2}{\sigma_0} \left(\frac{\gamma_w}{K_{an}} \right)^4 ((\vec{\nabla} \wedge [\nu])[V^2])^{\odot 2} + \vec{U}_{ex} \frac{J_{\sigma}^2}{\sigma_0} \left(\frac{\gamma_w}{K_{an}} \right)^4 ([\nu]([\vec{\nabla}] \wedge [V^2]))^{\odot 2}\end{aligned}\quad (29)$$

squared time derivative of $[V^2]$ that involves a time delay parameter τ of domains structuring, related to the walls' mobility with the conductivity and permeability inside the domains.

$$\vec{\mathcal{H}}_{ed} \propto \frac{1}{2} \frac{\gamma_w^4}{K_{an}^3} \tau \partial_t [V^2] \odot \tau \partial_t [V^2] \quad (22)$$

$$\vec{H} \cdot \partial_t \vec{B} \approx [\nu][V^2] \cdot \vec{\varphi} \partial_t [V^2] \cdot \vec{\varphi} + \underbrace{[\nu] \tau_b \partial_t [V^2] \cdot \vec{\varphi} \cdot \partial_t [V^2] \cdot \vec{\varphi}}_{\vec{\mathcal{H}}_{ed} \propto [\nu] \tau_b \partial_t [V^2] \odot \tau_b \partial_t [V^2]} \quad (30)$$

The rest of the Hamiltonian allows to write the governing diffusion

equation (31) for $\vec{\varphi}$.

$$\delta \int \left([\nu][V^2] \left(\sigma^{-1}[\nu][V^2] |\vec{\nabla} \wedge \vec{\varphi}|^2 + \vec{\varphi} \cdot \partial_t [V^2] \vec{\varphi} \right) \right) dt = 0 \quad (31)$$

This suggests that solving the governing equations of the TMPT should implicitly solve part of the Maxwell equations, the one that depends on the Material heterogeneous structure, for which both the micromagnetic and electromagnetic theories govern the mechanisms. The entanglement and connection between the classical energy contributions of electromagnetic fields and the TMPT Lagrangian also provide the relationship between the intelligible physical properties like γ_w and K_{an} and the so called magnetic reluctivity $[\nu]$ or like the time delays τ and τ_b . Additionally, this paper investigates how to derive the domains structuring and magnetization laws through the Lagrange principle [31] and the corresponding conservation laws with invariants linked to the Nøther theorem [32].

Derivation with the principle of least vectorial Action

Quasi-Static conservative formulation (Euler Lagrange equations)

The vectorial action $\vec{\mathcal{S}}$ (32) of the TMPT and its components (33) can be classically defined by the space and time integral of $\vec{\mathcal{L}}$:

$$\vec{\mathcal{S}} = \iiint \int \vec{\mathcal{L}}([V^2], \partial_t[V^2], t) dt d^3_x \quad (32)$$

$$\mathcal{S}_j = \iiint \int \mathcal{L}_j(V_{ij}^2, \partial_t V_{ij}^2, t) dt d^3_x \quad (33)$$

Applying then the principle of stationary action (34), vectorial action here, with no explicit time variation of the Lagrangian $\vec{\mathcal{L}}$; it is possible to derive, through the Euler Lagrange equations (tensor equation in this case), the tensor governing equation for $[V^2]$ (35) (similar to a Poisson-like equation [31]).

$$\begin{aligned} \delta \vec{\mathcal{S}} = 0 &\Leftrightarrow \delta \mathcal{S}_j = 0 \\ &\Leftrightarrow \left(\frac{\partial \mathcal{L}_j}{\partial V_{ij}^2} - \partial_t \left(\frac{\partial \mathcal{L}_j}{\partial \partial_t V_{ij}^2} \right) \right) = 0 \\ &\Leftrightarrow (f \ll GHz) \left(- \frac{\partial \mathcal{L}_j}{\partial V_{ij}^2} \right) = 0 \\ &\Leftrightarrow (f \ll GHz) [\Delta][V^2] \\ &\quad - \frac{C_{an}}{C_{ex}} \left(\frac{K_{an}}{\gamma_w} \right)^2 \left([V^2] - \left(\frac{\vec{K}_{an} \cdot ([V^2] \cdot \vec{K}_{an})^T}{|\vec{K}_{an}|^2} \right)^T \right) \\ &\quad + \frac{3}{2} \frac{C_{\lambda s}}{C_{ex}} \left(\frac{K_{an}}{\gamma_w} \right)^2 \left(\frac{[\lambda] \cdot [s]}{K_{an}} \cdot [V^2] \right) \\ &\quad - \frac{9}{4} \frac{C_{\lambda \varepsilon}}{C_{ex}} \left(\frac{K_{an}}{\gamma_w} \right)^2 \left(\frac{[\lambda^2] \cdot [C]}{K_{an}} \cdot [V^2] \right) = [0] \end{aligned} \quad (34)$$

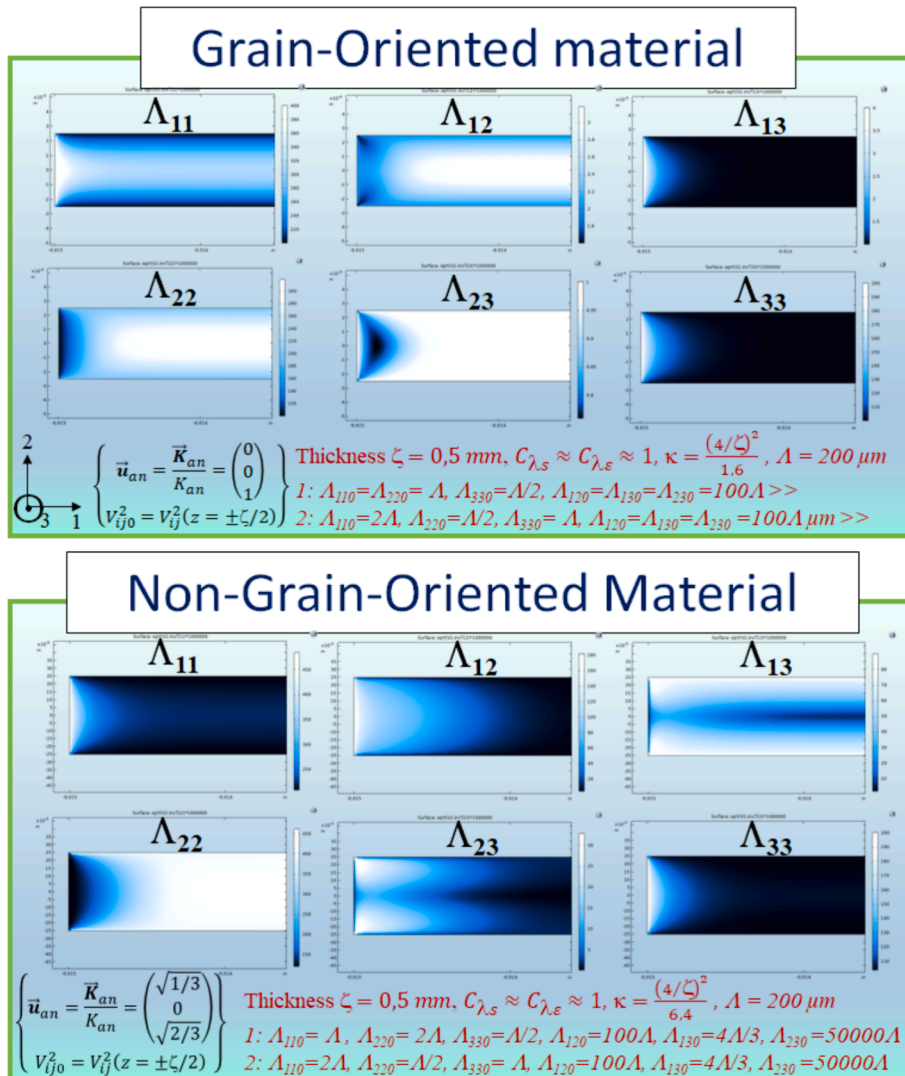


Fig. 3. FEM computation of the quasi-static TMPT equations and results obtained on $[\Lambda]$ components within the cross section of electrical steel sheets with thickness ζ and boundary conditions fixed ($\kappa = \kappa_{an} + \kappa_{\lambda \varepsilon} - \kappa_{\lambda \sigma} = (C_{an}/C_{ex}) * (K_{an}/\gamma_w)^2 + (C_{\lambda \varepsilon}/C_{ex}) * (K_{an}/\gamma_w)^2 (9\lambda^2 C/4K_{an}) - (C_{\lambda s}/C_{ex}) * (K_{an}/\gamma_w)^2 (3\lambda s/2K_{an})$).

Fig. 3 shows one example of a result we can achieve by implementing such governing equations with the Finite Element Method (FEM) to find the magnetic structure from the point of view of tensor magnetic phases in the cross section of an electrical steel sheet, either grain oriented or non grain oriented. The figure gives the components of tensor $[\Lambda]$, which is defined by $[\Lambda^2] = [\Lambda]^2 = [V^2]^{-1}$. What is remarkable is that it seems possible to overlook the gradients of domains types and sizes from the surfaces (prioritizing closure domains) towards the volume (with importance given to the main domains oriented as a function of the anisotropy of the material). However this formulation still requires to know the boundary conditions as previously mentioned and given in the figure legend.

Dynamic dissipative formulation (Nøther equations)

Still applying then the principle of stationary vectorial action, but adding a possible time variation of $\vec{\mathcal{L}}$ (decreasing due to dissipation) and using the Nøther equations (36) [32]; it is possible to derive, through the Euler Lagrange Eqs. (37), the tensor dynamic diffusion-like Eq. (38) for $[V^2]$.

$$\delta \vec{\mathcal{L}} = 0 \Leftrightarrow \delta \mathcal{L}_j = 0$$

$$\Leftrightarrow \left(\frac{\partial \mathcal{L}_j}{\partial V_{ij}^2} - \partial_t \left(\frac{\partial \mathcal{L}_j}{\partial \partial_t V_{ij}^2} \right) \right) \delta V_{ij}^2 + \frac{d}{dt} \left(\frac{\partial \mathcal{L}_j}{\partial \partial_t V_{ij}^2} \delta V_{ij}^2 \right) + \frac{\partial \mathcal{L}_j}{\partial t} \delta t = 0 \quad (36)$$

$$\Leftrightarrow_{(f \ll GHz)} \frac{\partial \mathcal{L}_j}{\partial t} = - \frac{\partial \mathcal{W}_j}{\partial V_{ij}^2} \frac{\partial V_{ij}^2}{\partial t} = + \frac{\partial \mathcal{R}_{edj}}{\partial \tau \partial_t V_{ij}^2} \frac{\partial V_{ij}^2}{\partial t}$$

$$\Leftrightarrow \left(\frac{\partial \mathcal{W}_j}{\partial V_{ij}^2} + \frac{\partial \mathcal{R}_{edj}}{\partial \tau \partial_t V_{ij}^2} \right) = 0 \quad (37)$$

$$\Leftrightarrow_{(f \ll GHz)} [\Delta][V^2]$$

$$\frac{C_{an}}{C_{ex}} \left(\frac{K_{an}}{\gamma_w} \right)^2 \left([V^2] - \frac{\left(\vec{K}_{an} \cdot ([V^2] \cdot \vec{K}_{an})^T \right)^T}{|\vec{K}_{an}|^2} \right)$$

$$+ \frac{3}{2} \frac{C_{is}}{C_{ex}} \left(\frac{K_{an}}{\gamma_w} \right)^2 \left(\frac{[\lambda] \cdot [s]}{K_{an}} \cdot [V^2] \right)$$

$$- \frac{9}{4} \frac{C_{\lambda e}}{C_{ex}} \left(\frac{K_{an}}{\gamma_w} \right)^2 \left(\frac{[\lambda^2] \cdot [C]}{K_{an}} \cdot [V^2] \right)$$

$$- \frac{1}{C_{ex}} \left(\frac{K_{an}}{\gamma_w} \right)^2 \tau \partial_t [V^2] = [0] \quad (38)$$

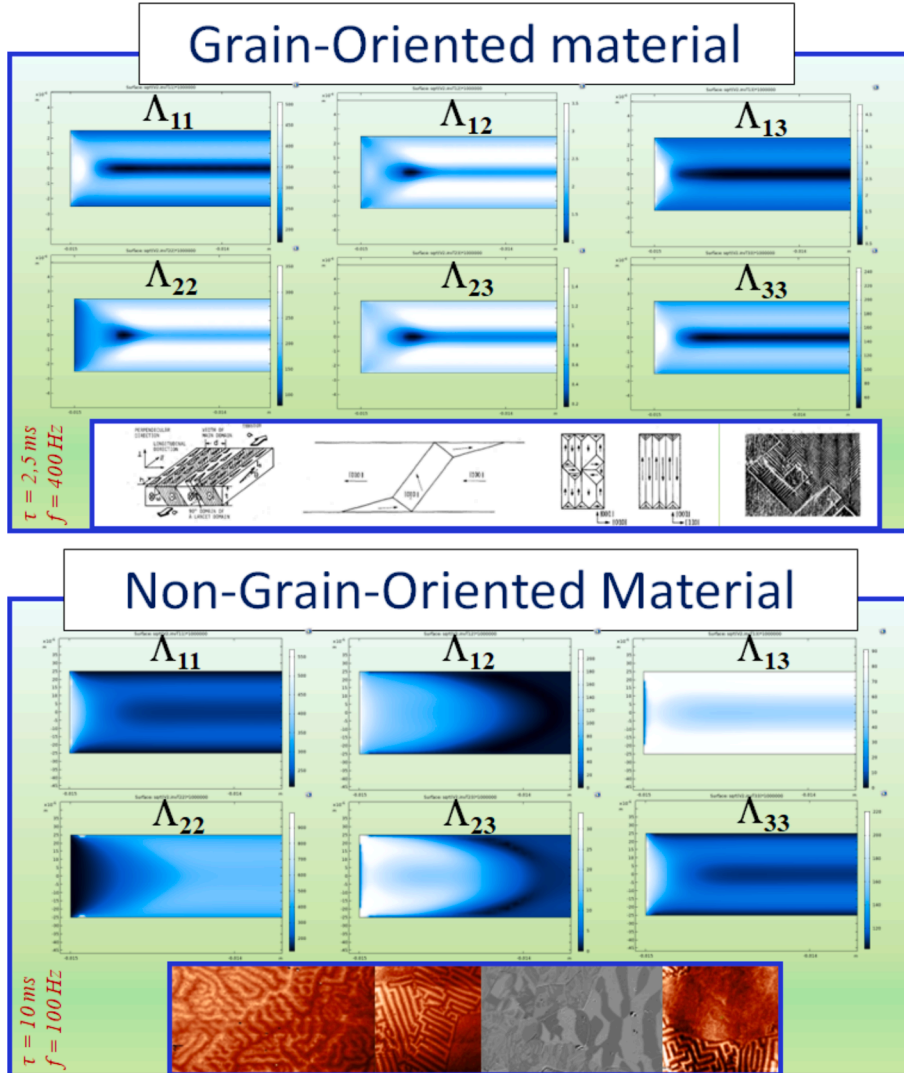


Fig. 4. FEM computation of the dynamic TMPT equations and results obtained on $[\Lambda]$ components within the cross section of electrical steel sheets with thickness ζ , boundary conditions fixed and magneto-harmonic approximations (sinusoidal waveforms at frequency $f = 400$ Hz) (some of GO figures come from J.W. Shilling and J.R.L. Houze, Trans. On Mag., vol. MAG-10, o. 2, pp. 195, 1974 and M. Inamura, IEEE Trans. Mag, vol. MAG-19, no. 1, 1983.).

Using the same test case of the previous § (Fig. 3), we can compute the dynamic Partial Differential Equations (PDE) with the Finite Element Method (FEM) and find the magnetic structure from the point of view of the tensor magnetic phases in the cross section of an electrical steel sheet, either grain oriented or non-grain oriented. In Fig. 4, it seems possible to overlook the additional gradients and delays of domains types and sizes due to the anti eddy field that can prevent the surface domains from moving, deforming and multiplying like the ones in the volume. Finally, the TMPT must be discussed while checking its coherence when considering reversal CPT symmetries [35] and invariance necessity [33,34].

Invariance and symmetry analysis

According to H. Weyl [33], if nature was entirely describable by laws, the whole phenomena would show total symmetry of universal laws, like they are formulated in the theory of relativity [37]. The fact that such symmetry is not apparent everywhere proves that contingency is an essential characteristic of the world. A contraposition states that a formulation that does not satisfy any invariance property cannot be a natural law. Reciprocally, satisfying such conditions does not guarantee any physical sense. According to E. Wigner then [34], The progression from events to laws of nature and from laws of nature to symmetry or/and invariance principles, is what to be meant by the hierarchy of our knowledge of the world around us. So we should look at some important symmetry and invariance properties, reasonably expected for any physical science theory that has the ambition to describe part of nature by laws.

Charge, parity and time reversal symmetry

The CPT symmetry property is a fundamental symmetry of physical laws under the transformations, potentially simultaneous transformations, of Charge conjugation (C: flip in the sign of a charge, a dipole moment or a current), Parity transformation (P: flip in the sign of one spatial coordinate. In three dimensions, it can also refer to the simultaneous flip in the sign of all three spatial coordinates, i.e. a point reflection) and Time reversal (T: flip in the sign of time). The CPT theorem [35] says that CPT symmetry should hold for all physical phenomena, or more precisely, that any invariant theory formulation with a Hermitian Hamiltonian must have CPT symmetry.

This theorem raises the question regarding which statement of the theory, which formulation, must satisfy the CPT symmetry: the variables $[\Lambda^2]$ and $\vec{\varphi}$ ((5) and (6)), the Lagrangian $\vec{\mathcal{L}}$ or/and action $\vec{\mathcal{S}}$ and Hamiltonian \mathcal{H} energy densities ((11–22) and (28–31)) or the Partial Differential Equations ((35), (38), (31)).

Charge conjugation (C):

Let’s consider first the Charge conjugation which leads to magnetization reversal: $\vec{J}_s = J_s \vec{1} \xrightarrow{C} -\vec{J}_s = J_s -\vec{1}$. Following the definitions (5) and (6), we deduce the following symmetry (39–40) for the variables:

$$[\Lambda^2] \xrightarrow{C} [\Lambda^2] \tag{39}$$

$$\vec{\varphi} \xrightarrow{C} -\vec{\varphi} \tag{40}$$

As a consequence, the charge conjugation keeps invariant the formulation of either the TMPT variable $[\Lambda^2]$, its Lagrangian $\vec{\mathcal{L}}$ and Action $\vec{\mathcal{S}}$, the PDE given by (35) and (38), but also the MAXWELL Hamiltonian \mathcal{H} or (31) and the corresponding diffusion equation for $\vec{\varphi}$.

Parity reflection (P):

Let’s consider then the space reflection transformation which leads to: $\vec{x} \xrightarrow{P} -\vec{x}$ and $\vec{v} \xrightarrow{P} -\vec{v}$. Following the definitions (5) and (6), we

Table 1

CPT symmetry properties of the TMPT theory.

	Tensor Magnetic Phase $[\Lambda^2]$ formulation				Flux $\vec{\varphi}$ formulation		
	$[\Lambda^2]$	$\vec{\mathcal{L}}$	$\vec{\mathcal{S}}$	PDE	$\vec{\varphi}$	\mathcal{H}	PDE
C	⊗	⊗	⊗	⊗	–	⊗	⊗
P	⊗	⊗	⊗	⊗	⊗	⊗	⊗
T	⊗	⊗	⊗	∅	⊗	∅	∅
CP	⊗	⊗	⊗	⊗	–	⊗	⊗
PT	⊗	⊗	⊗	∅	⊗	∅	∅
CT	⊗	⊗	⊗	∅	–	∅	∅
CPT	⊗	⊗	⊗	∅	–	∅	∅

deduce the following symmetry (41–42) for the variables:

$$[\Lambda^2] \xrightarrow{P} [\Lambda^2] \tag{41}$$

$$\vec{\varphi} \xrightarrow{P} \vec{\varphi} \tag{42}$$

Therefore, the space reflection keeps invariant the formulation of either the TMPT variable $[\Lambda^2]$, its Lagrangian $\vec{\mathcal{L}}$ and Action $\vec{\mathcal{S}}$, the PDE given by (31), (35) and (38), but also the MAXWELL Hamiltonian \mathcal{H} or (31) and the corresponding diffusion equation for $\vec{\varphi}$.

Time reversal (T):

Let’s consider finally a time reversal which leads to: $t \xrightarrow{T} -t$ and $\partial_t \xrightarrow{T} -\partial_t$. Following the definitions (5) and (6), we deduce the following symmetry (43–44) for the variables:

$$[\Lambda^2] \xrightarrow{T} [\Lambda^2] \tag{43}$$

$$\vec{\varphi} \xrightarrow{T} \vec{\varphi} \tag{44}$$

The time reversal keeps invariant the formulation of either the variable $[\Lambda^2]$, its Lagrangian $\vec{\mathcal{L}}$ and Action $\vec{\mathcal{S}}$, but also the PDE given by (35). However, the MAXWELL Hamiltonian \mathcal{H} or (31) and the diffusion-like PDE given by (31) and (38) do not satisfy any T symmetry because of its intrinsic irreversible character.

CPT symmetry properties of the TMPT theory are summed up in Table 1.

Principle of invariance with respect to a change of reference frame

According to E. Wigner [33,34], it is not necessary to look deeper into the situation to realize that laws of nature could not exist without principles of invariance. If the correlations between events changed from day to day, it would be different at different points of space and it would be impossible to discover them. Thus, the invariance of the laws of nature with respect to displacements in space and time are almost a necessary prerequisite to discovering, or even cataloguing, the correlations between events which are the laws of nature. The most famous group of transformations that keeps invariant the laws of nature is the Lorentz Group [36,37] widely used in the specific Relativity theory [38].

When it comes to the electromagnetic fields in matter with constitutive relationships like $\vec{D} = \epsilon \vec{E}$ and $\vec{H} = \nu \vec{B}$. There is a controversy whether the electromagnetic fields \vec{E} , \vec{B} , \vec{D} , \vec{H} can be combined or not in relativistic tensors if \vec{D} and \vec{H} are related to \vec{E} and \vec{B} by the linear constitutive relationships. Recently, the author J. Franklin [39] went to the conclusion that there is no way in special relativity to calculate the electromagnetic fields in any case where there is a moving linearly polarizable medium. Even calculating the \vec{E} and \vec{B} fields in the rest system of the polarizable medium, and then trying to Lorentz transforming the fields to a moving system would not work because the \vec{E}

and \vec{B} fields themselves cannot be combined in a relativistic tensor in the presence of a polarizable medium. Some method other than the Lorentz transformation would have to be used to treat the electromagnetism of a moving polarizable medium. For instance, non-relativistic treatment to 1st order in the velocity could be used.

So, it is proposed to look for the group properties to give to a reference frame transformation towards a new frame with Galilean motion at speed \vec{v} , below the speed of light c , that satisfy the invariance principle for the space integral of Lagrangian $\vec{\mathcal{L}}$ (11) with dissipation $\vec{\mathcal{H}}_{ed}$ (22) and its Action $\vec{\mathcal{S}}$ (32). \vec{v} refers to the speed of motion of the magnetic system as a whole. Basic linear calculations led to a group of transformations named \mathbb{V} with the property $\det \mathbb{V} = 1$ (refer to appendix 5). Each transformation \mathbb{V} involves the Lorentz-like factors $\vec{\beta}$ and α (45), which look like the ones of relativity [36–38] but with two important and significant differences (ϑ instead of c and the sign + instead of -).

$$\vec{\beta} = \frac{\vec{v}}{\vartheta} \text{ and } \alpha = \left(1 + |\vec{\beta}|^2\right)^{-1/2} \quad (45)$$

The speed v can be higher than the property ϑ of (46), which is not a fundamental constant but represents the magnetic ordering speed, i.e. the exchange length [11,19] over the time delay of domains structuring.

$$\vartheta = \sqrt{2C_{ex}} \frac{\gamma_w}{K_{an}\tau} = \frac{M_s}{\tau} \sqrt{\frac{2A_{ex}}{K_{an}}} \quad (46)$$

Transformations \mathbb{V} are represented by a 4*4 matrix (47) similar to that of a dual 3D Lorentz transformation [36–38]. It can be simplified by a typical 4*4 matrix (48) ($\beta = \beta_1$) still very similar to 1D Lorentz matrices.

$$\mathbb{V} = \begin{bmatrix} \alpha & -\alpha\beta_1 & -\alpha\beta_2 & -\alpha\beta_3 \\ +\alpha\beta_1 & 1 + (\alpha - 1)\frac{\beta_1^2}{\beta^2} & (\alpha - 1)\frac{\beta_1\beta_2}{\beta^2} & (\alpha - 1)\frac{\beta_1\beta_3}{\beta^2} \\ +\alpha\beta_2 & (\alpha - 1)\frac{\beta_2\beta_1}{\beta^2} & 1 + (\alpha - 1)\frac{\beta_2^2}{\beta^2} & (\alpha - 1)\frac{\beta_2\beta_3}{\beta^2} \\ +\alpha\beta_3 & (\alpha - 1)\frac{\beta_3\beta_1}{\beta^2} & (\alpha - 1)\frac{\beta_3\beta_2}{\beta^2} & 1 + (\alpha - 1)\frac{\beta_3^2}{\beta^2} \end{bmatrix} \quad (47)$$

$$\mathbb{V}(\vec{v} = \vec{v}_1) = \begin{bmatrix} \alpha & -\alpha\beta & 0 & 0 \\ +\alpha\beta & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (48)$$

Then the metrics to be used in association with this group of transformation is defined by the metric tensor $g_{\mu\nu} = \text{diag}(1, 1, 1, 1)$ and means that 4-vectors invariant through the \mathbb{V} transformations can be defined by (49) for the 4-position, (50) for the velocity, (51) for the force and (52) for the current density.

$$(\vartheta t; \vec{x}) = (\vartheta t, x_1, x_2, x_3)^t \quad (49)$$

$$\alpha(\vartheta; \vec{v}) \quad (50)$$

$$\alpha(\vec{\beta} \cdot \vec{F}; \vec{F}) \quad (51)$$

$$\alpha\left(\frac{\vartheta}{\sigma}; \vec{j}\right) \quad (52)$$

This invariance principle simply tells us that the physical laws of the TMPT represented by the general Lagrangian (11) with dissipation $\vec{\mathcal{H}}_{ed}$ (22) and its Action (32) can be invariant through a Galilean motion of the main reference frame at speed \vec{v} providing a \mathbb{V} transformation of the

4-vector $(\vartheta t, \vec{x})$ and the space and time differential operators. According to [40] at the beginning of the 20th century, the invariance theory is a relative concept, because it is possible to use one different invariance theory for each group of transformation. It becomes so natural to find a group of transformation \mathbb{V} in the case of the TMPT theory which is different from the invariance principle exposed in the well known theory of relativity [38].

Applications of the tensor magnetic Phase theory

Ferromagnetic materials

The TMPT and any model able to describe the domains structure of a material can be used for ferromagnetic materials for which the hysteresis loops, apparent permeability and losses are closely related to the magnetization mechanisms within domains and walls. One of challenges then is still to describe the geometry dependent scalar but also vectorial behaviour of such materials [41,42]. The quasi-static 1D TMPT implementation for a NGOs first [43,44] and GOES then [28], with various thicknesses, giving its apparent permeability and iron losses with good approximation, has been one of the first encouraging results for the development of the theory within the dynamic condition in the volume too.

Ferroelectric and Multiferroic materials

The TMPT can be used not only for ferromagnetic polarized materials (index J) but also for Ferroelectric (index P) (containing electrically polarized domains) and multiferroic materials (containing both magnetically and electrically polarized domains). In this purpose, the electric polarization \vec{P} (54) has to be detailed next to the magnetic polarization (53), with probable relationships between the two.

$$\vec{J} = [V_J^2] \vec{\varphi}_J = [V_J^2] ([\Lambda_{J1}^2] - [\Lambda_{J1}^2]) J_s \vec{1} \quad (53)$$

$$\vec{P} = [V_P^2] \vec{\varphi}_P = [V_P^2] ([\Lambda_{P1}^2] - [\Lambda_{P1}^2]) P_s \vec{1} \quad (54)$$

Processes with TMPT control

One of the first industrial applications of this work is the assistance required and given to some manufacturing processes of material through a TMPT control. As an example, it is worth mentioning the investigations and developments performed to specify a pulsed laser surface treatment useful to act on the surface magnetic structure and therefore also inside the volume of an electrical steel. Different pulse durations (Long Pulsed Laser LPL, Short Pulsed Laser SPL and Ultra Short Pulsed Laser USPL) [45], powers and patterns [46] have been investigated to achieve the best possible control of beneficial magnetic structure (images through dedicated microscope called MOKE or MOIF [3]). Purpose of this magnetic structure control is to reduce both the iron losses up to 50 % in some configurations and the magnetic induced vibration responsible for the noise up to 20 dB.

Magnetic cores

Regarding the products manufacturing, we refer to a large number of magnetic cores, using ferromagnetic materials:

- The Ones with various shapes for which we may be interested in the permeability spectra
- The Ones inside the classical magnetic components such as the chokes, transformers, the motors, generators ... for which we are interested in the impedance, losses and yield

- The Ones inside accurate sensors and sensitivity actuators, requiring a high sensitivity not altered by the boundary and transient effects ... like in circuit breakers for examples

The main stake is definitely to ease the computation of fields diffusion inside the polarizable media inside the cores while avoiding the tremendous complexity and enormous time consumption of a direct coupling between electromagnetism and micro-magnetism [47], which would be impossible for a whole prototype.

Discussion, conclusion and forthcoming

The purpose of a theory in Physics is to be able to describe the reality or what we can observe with consistency. That is why this theory must be confronted with both observations and measurements.

Theory against observations

Firstly considering observations, it is not always possible to have access to the true magnetic structure inside the volume of samples larger than some nano or micrometers ... However, previous works, such as A. Hubert and R. Schafer in Germany [3], allowed us to compare what has been discovered by the images at disposal and what can be predicted by the TMPT. In a GOES for example, the tensor terms seem able to overlook again both the main 180° domains without gradients and the closure domains with gradients [28] (the Lancet domains, 90° domains, tree patterns etc.). The space variations are driven by the ratio κ between the anisotropy and the exchange. Nevertheless, the TMPT can certainly overlook the typical magnetic phases of domains in terms of statistical directions and sizes and its gradient into a material, but it cannot give the exact microscopic shape and arrangement of domains and walls.

Theory against measurements

Considering now measurements, the 1D TMPT problem has been calculated analytically to compute the dynamic magnetization of a GOES sheet sample in the rolling direction with various thicknesses. The two measurable observables named apparent permeability and losses have been found [28]. The TMPT result seems promising, since providing an accurate identification of the material properties (κ , related to the anisotropy over exchange plus magnetostriction; the boundary

surface structure Λ_0 and the time delay τ) for the domains structuring, it gives a perfect fitting of the apparent permeability and losses for the whole thicknesses, in reference [28] at low induction levels (linear assumptions).

Conclusion and forthcoming

The physics result of this work is an invariant energy formulation dedicated to polarized media such as ferromagnetic, ferroelectric and multiferroic materials. Results take into account the geometry, field and stress dependence of polarized domains and flux at a long distance, the variability of those polarized domains with phases, dynamics and losses, the anisotropic behaviour intrinsic in the tensor variables. Still work has to be done to develop the TMPT theory and formulation with non linear coupling to classical fields at boundaries and in the volume, variations with the whole kinds of exchange and anisotropy, rebuilding vector hysteresis loops with memory and dynamics, investigations on dual wave equations at high frequencies (Barkhausen noise, spin waves etc.).

CRedit authorship contribution statement

Olivier Maloberti: Formal analysis, Conceptualization, Data curation, Funding acquisition, Investigation, Methodology, Software, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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APPENDIX 1: Mathematical Tools, properties and specific TMPT operators

Operators.

$$[Y] = \begin{bmatrix} \vec{Y}_1 & \vec{Y}_2 & \vec{Y}_3 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix}$$

$$[n] = [\vec{n}, \vec{n}, \vec{n}]$$

$[Y]^T$ means to transpose the matrix $[Y]$.

$$[\vec{\nabla}] (\vec{A}) = [\vec{\nabla} \otimes \vec{A}], [\vec{\nabla}] (\vec{A})_{ij} = [\vec{\nabla} \otimes \vec{A}]_{ij} : \text{tensor gradient on a vector,}$$

$$\vec{\nabla} \cdot [Y] = \left(\vec{\nabla} \cdot \vec{Y}_1, \vec{\nabla} \cdot \vec{Y}_2, \vec{\nabla} \cdot \vec{Y}_3 \right)^T : \text{vector divergence on a tensor,}$$

$$[\vec{\nabla}] \times [Y] = [\vec{\nabla} \times [Y]] = \left[\vec{\nabla} \wedge \vec{Y}_1, \vec{\nabla} \wedge \vec{Y}_2, \vec{\nabla} \wedge \vec{Y}_3 \right]^T : \text{tensor rotational on a tensor,}$$

$$[\Delta][Y] = \left[\vec{\Delta} (\vec{Y}_1), \vec{\Delta} (\vec{Y}_2), \vec{\Delta} (\vec{Y}_3) \right]^T : \text{tensor Laplacian on a tensor,}$$

$$\vec{\nabla} \cdot = (\partial_1, \partial_2, \partial_3)^T : \text{Nabla operator,}$$

$$\vec{\Delta} \cdot = (\partial_1 \partial_1, \partial_2 \partial_2, \partial_3 \partial_3)^T : \text{Laplacian operator.}$$

Operations.

$[\vec{A} \otimes \vec{B}]$, $[\vec{A} \otimes \vec{B}]_{ij} = A_i B_j$: tensor product between two vectors.

$[[Y] \cdot [Z]]$, $[[Y] \cdot [Z]]_{ij} = Y_{ij} Z_{ij}$: tensor product type 1 between 2 tensors.

$[[Y] \times [Z]] = \left[\vec{Y}_1 \wedge \vec{Z}_1, \vec{Y}_2 \wedge \vec{Z}_2, \vec{Y}_3 \wedge \vec{Z}_3 \right]^T$: tensor product type 2 between 2 tensors.

$([Y] \odot [Z]) = \left(\vec{Y}_1 \cdot \vec{Z}_1, \vec{Y}_2 \cdot \vec{Z}_2, \vec{Y}_3 \cdot \vec{Z}_3 \right)^T$, $([Y] \overset{\circ}{\odot} [Z]) = ([Y] \odot [Z])$: vector product between 2 tensors.

$[Y]^\circ [Z] = Y_{ij} Z_{ij}$: scalar product between two tensors.

Derivative and integral properties

$$[\vec{\nabla} \times][\vec{\nabla} \times][Y] = [\vec{\nabla}](\vec{\nabla} \cdot [Y]) - [\Delta][Y]$$

$$[X] \odot ([Y] \times [Z]) = [Z] \odot ([X] \times [Y]) = [Y] \odot ([Z] \times [X])$$

$$\vec{\nabla} \cdot ([Y] \vec{A}) = (\vec{\nabla} \cdot [Y]) \cdot \vec{A} + [Y]^\circ [\vec{\nabla} \otimes \vec{A}]$$

$$\iint_{\Omega} (\vec{\nabla} \cdot \vec{A}) d^3x = \oint_{\partial\Omega} (\vec{A} \cdot \vec{n}) d^2x$$

$$\iint_{\Omega} (\vec{\nabla} \wedge \vec{A}) d^3x = \oint_{\partial\Omega} (\vec{A} \wedge \vec{n}) d^2x$$

$$\iint_{\Omega} (\vec{\nabla} \cdot [Y]) d^3x = \oint_{\partial\Omega} ([Y] \odot [n]) d^2x$$

$$\iint_{\Omega} ([\vec{\nabla}] \times [Y]) d^3x = \oint_{\partial\Omega} ([Y] \times [n]) d^2x$$

$$\iint_{\Sigma} ([\vec{\nabla}] \times [Y]) d^2x = \oint_{\partial\Sigma} ([Y] \odot [n]) dx$$

APPENDIX 2: Origin and formulation of the magnetic exchange energy

The Exchange energy has got a quantum origin at the atomistic and microscopic scales, using the following physical properties:

γ : the gyromagnetic factor $[C.kg^{-1}] = [s^{-1}.T^{-1}]$.

J_{ex} : the quantum Exchange integral Energy $[J]$.

$dV = d\Omega_{lattice} = dx_1 dx_2 dx_3$: volume of each crystal lattice $[m^3]$.

dx_1, dx_2, dx_3 : geometrical dimensions of one crystal lattice $[m]$.

Z_c : the coordination number or ligancy of each atom of the crystal lattice.

$\vec{m}_i = \frac{d\vec{\mu}_i}{dV} = \left(\frac{g\mu_B}{\hbar} \right) \frac{d\vec{s}_i}{dV} = (\gamma) \frac{d\vec{s}_i}{dV}$: the microscopic magnetization \vec{m}_i defined by the volume density of magnetic moments $\vec{d\mu}_i$ $[A.m^2]$ carried by the atoms and due to the density of electrons' spin momentum \vec{ds}_i $[J.s]$.

The original microscopic Exchange energy ξ_{ex} $[J]$ comes from a discrete summation of the interactions of every couples (i,j) of atomic spins momentum of electrons \vec{ds}_i and \vec{ds}_j .

$$\xi_{ex} = \sum_i \sum_j - \frac{J_{ex}}{\hbar^2} \vec{ds}_i \cdot \vec{ds}_j = \sum_i \sum_j - \frac{J_{ex} dV^2}{(g\mu_B)^2} \vec{m}_i \cdot \vec{m}_j = \sum_i \sum_j - \frac{J_{ex} dV^2}{(\gamma\hbar)^2} \vec{m}_i \cdot \vec{m}_j$$

Each microscopic magnetization vector \vec{m}_j may be expanded in a Taylor series until the second order, as a function of its neighboring microscopic magnetization vector \vec{m}_i :

$$\vec{m}_j \approx \vec{m}_i + \left(\partial_{x_{kj}} \vec{m}_i dx_{kj} \right) + \frac{1}{2} \left(\partial_{x_{kj}}^2 \vec{m}_i dx_{kj}^2 \right) + \theta(dx_{kj}^2)$$

On summing only over nearest neighbors of each atom i:

$$\xi_{ex} \approx \sum_{\substack{1^{st} \text{ neighbors} \\ 2^{nd} \text{ order}}} \sum_i \sum_{j=1}^{Z_c} - \frac{J_{ex} dV^2}{(\gamma\hbar)^2} \left(m_i^2 + \vec{m}_i \cdot \left(\partial_{x_{kj}} \vec{m}_i dx_{kj} \right) + \frac{1}{2} \vec{m}_i \cdot \left(\partial_{x_{kj}}^2 \vec{m}_i dx_{kj}^2 \right) \right)$$

Let's develop the final approximate expression of the variable part of the Exchange Energy for a body-centered cubic lattice with lattice constant a :

$$\xi_{ex} \underset{\substack{\approx \\ 1^{st} \text{ neighbors } 2^{nd} \text{ order}}}{\approx} \sum_i -A_{ex} \left(\vec{m}_i \bullet \sum_{k=1}^3 \left(\partial_{x_k}^2(\bullet) \right) \vec{m}_i \right) dV = \sum_i -A_{ex} (\vec{m}_i \bullet \Delta(\vec{m}_i)) dV$$

With $A_{ex} = \frac{J_{ex} dV^2}{(\gamma h)^2 a}$ the macroscopic Exchange coefficient [$J.m.A^{-2}$]

and $\Delta(\bullet) = \vec{\nabla}^2(\bullet) = \sum_{k=1}^3 \partial_{x_k}^2(\bullet)$ the Laplacian operator in cartesian coordinates.

The fact that each magnetization vector has got a constant modulus given by the saturation magnetization $\vec{m}_i \bullet \vec{m}_i = M_s^2$, we can use the following relationships:

$$0 = \partial_{x_k} (M_s^2) = \partial_{x_k} (\vec{m}_i \bullet \vec{m}_i) = 2 \cdot \vec{m}_i \bullet \partial_{x_k} (\vec{m}_i) \Rightarrow \\ 0 = \partial_{x_k} (\vec{m}_i \bullet \partial_{x_k} (\vec{m}_i)) = (\partial_{x_k} (\vec{m}_i))^2 + \vec{m}_i \bullet \partial_{x_k}^2 (\vec{m}_i) \Leftrightarrow -\vec{m}_i \bullet \partial_{x_k}^2 (\vec{m}_i) = (\partial_{x_k} (\vec{m}_i))^2$$

Then the Exchange energy can be written by two ways like in the literature:

$$\xi_{ex} \underset{\substack{\approx \\ 1^{st} \text{ neigh. } 2^{nd} \text{ order}}}{\approx} \sum_i -A_{ex} (\vec{m}_i \bullet \Delta(\vec{m}_i)) dV = \sum_i +A_{ex} \sum_{k=1}^3 (\partial_{x_k} (\vec{m}_i))^2 dV = \sum_i +A_{ex} \sum_{k=1}^3 (\vec{\nabla}(m_{ik}))^2 dV$$

$$\text{By using the equivalence: } \begin{cases} \sum_{k=1}^3 (\partial_{x_k} (\vec{m}_i))^2 \\ = (\partial_{x_1} (\vec{m}_i))^2 + (\partial_{x_2} (\vec{m}_i))^2 + (\partial_{x_3} (\vec{m}_i))^2 \\ = \sum_{k=1}^3 \left((\partial_{x_1} (m_{ik}))^2 + (\partial_{x_2} (m_{ik}))^2 + (\partial_{x_3} (m_{ik}))^2 \right) = \sum_{k=1}^3 (\vec{\nabla}(m_{ik}))^2 \\ \neq (\vec{\nabla} \bullet \vec{m}_i)^2 = \left(\sum_{k=1}^3 \partial_{x_k} (m_{ik}) \right)^2 \end{cases}$$

Considering a very large number of atoms into the material, the crystal size being infinitely small compared to the sample size ($dV \rightarrow 0$), the discrete formulation tends towards its continuous integral formulation that follows:

$$\xi_{ex} \xrightarrow{dV \rightarrow 0} \iiint_V -A_{ex} (\vec{m} \bullet \Delta(\vec{m})) dV = \iiint_V +A_{ex} \sum_{k=1}^3 (\vec{\nabla}(m_{ik}))^2 dV$$

This gives a second order approximation of the Exchange energy density per unit volume μ_{ex} [$J.m^{-3}$]:

$$\mu_{ex} \underset{\substack{\approx \\ 2^{nd} \text{ order}}}{\approx} -A_{ex} (\vec{m} \bullet \Delta(\vec{m})) = +A_{ex} \sum_{k=1}^3 (\vec{\nabla}(m_k))^2$$

Another expression is possible thanks to the vector identity $\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{m} = \vec{\nabla}(\vec{\nabla} \bullet \vec{m}) - \Delta(\vec{m})$, the Green Ostrogradsky theorem $\iiint_V \vec{\nabla} \cdot \vec{T} dV = \oint_{\partial V} \vec{T} \cdot \vec{dS}$ and the Integration by parts:

$$\xi_{ex} \xrightarrow{dV \rightarrow 0} -A_{ex} \iiint_V (\vec{m} \bullet \Delta(\vec{m})) dV = +A_{ex} \iiint_V \vec{m} \bullet (\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{m} - \vec{\nabla}(\vec{\nabla} \bullet \vec{m})) dV$$

The first integral becomes (Einstein notation with the use of the Levi-Civita tensor¹):

$$\iiint_V \vec{m} \bullet (\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{m}) dV = \iiint_V (m_p \varepsilon_{pqr} \partial_q (\varepsilon_{rij} \partial_i m_j)) dV = \iiint_V \varepsilon_{pqr} \partial_q (m_p \varepsilon_{rij} \partial_i m_j) dV - \iiint_V (\varepsilon_{pqr} \partial_q (m_p) (\varepsilon_{rij} \partial_i m_j)) dV \\ = - \iiint_V \varepsilon_{qpr} \partial_q (m_p \varepsilon_{rij} \partial_i m_j) dV + \iiint_V (\varepsilon_{qpr} \partial_q m_p) (\varepsilon_{rij} \partial_i m_j) dV \\ = \iiint_V \vec{\nabla} \bullet (-\vec{m} \wedge \vec{\nabla} \wedge \vec{m}) dV + \iiint_V (\vec{\nabla} \wedge \vec{m}) \bullet (\vec{\nabla} \wedge \vec{m}) dV$$

¹ ε_{pqr} is the Levi-Civita tensor defined by: $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$ and $\varepsilon_{ppr} = \varepsilon_{ppq} = \varepsilon_{rpr} = 0$

$$= \oint_{\partial V} (-\vec{m} \wedge \vec{\nabla} \wedge \vec{m}) \cdot d\vec{S} + \iiint_V |\vec{\nabla} \wedge \vec{m}|^2 dV$$

The second integral gives:

$$\begin{aligned} \iiint_V \vec{m} \bullet (-\vec{\nabla}(\vec{\nabla} \bullet \vec{m})) dV &= \iiint_V -m_p \partial_p (\partial_q m_q) dV \\ &= \iiint_V -\partial_p (m_p (\partial_q m_q)) dV + \iiint_V (\partial_p m_p) (\partial_q m_q) dV \\ &= \iiint_V \vec{\nabla} \bullet (-(\vec{\nabla} \bullet \vec{m}) \vec{m}) dV + \iiint_V (\vec{\nabla} \bullet \vec{m}) \bullet (\vec{\nabla} \bullet \vec{m}) dV \\ &= \oint_{\partial V} (-(\vec{\nabla} \bullet \vec{m}) \vec{m}) \cdot d\vec{S} + \iiint_V |\vec{\nabla} \bullet \vec{m}|^2 dV \end{aligned}$$

Another vector identity can be used: $-\vec{m} \wedge \vec{\nabla} \wedge \vec{m} = -\frac{1}{2} \vec{\nabla}(\vec{m} \bullet \vec{m}) + (\vec{m} \bullet \vec{\nabla}) \vec{m} = (\vec{m} \bullet \vec{\nabla}) \vec{m}$

$$\vec{\nabla}(\vec{m} \otimes \vec{m}) = (\vec{\nabla} \bullet \vec{m}) \vec{m} + (\vec{m} \bullet \vec{\nabla}) \vec{m}$$

$$\text{Finally: } \xi_{ex} \xrightarrow{dv \rightarrow 0} \iiint_V A_{ex} |\vec{\nabla} \wedge \vec{m}|^2 dV + \iiint_V A_{ex} |\vec{\nabla} \bullet \vec{m}|^2 dV + \oint_{\partial V} A_{ex} (\vec{\nabla}(\vec{m} \otimes \vec{m}) - 2(\vec{\nabla} \bullet \vec{m}) \vec{m}) \cdot d\vec{S}$$

In this paper, we consider that any non-nil contribution of magnetic charges $(\vec{\nabla} \bullet \vec{m})$ in the volume of the material is necessarily due to a demagnetizing field. So this energy integral $\iiint_V A_{ex} |\vec{\nabla} \bullet \vec{m}|^2 dV$ shall be included in the demagnetizing energy, minimized when $\vec{\nabla} \bullet \vec{m} = \vec{\nabla} \bullet \left(\frac{\vec{b}}{\mu_0} - \vec{h} \right) \rightarrow \vec{\nabla} \bullet \left(\frac{\vec{b}}{\mu_0} \right) = 0$.

Similarly, the surface energy integral $\oint_{\partial V} A_{ex} (\vec{\nabla}(\vec{m} \otimes \vec{m}) - 2(\vec{\nabla} \bullet \vec{m}) \vec{m}) \cdot d\vec{S}$ contributes to the demagnetizing field energy due to the surface magnetic poles, taken for the energy minimization principle into account through an adequate surface boundary condition.

At the end, assuming the two previous assumptions we've got the following equivalence for the Exchange energy density per unit volume $\mu_{ex} [J \cdot m^{-3}]$ and the corresponding energy integral:

$$\begin{aligned} \mu_{ex} &\underset{2^{nd} \text{ order}}{\approx} -A_{ex} (\vec{m} \bullet \Delta(\vec{m})) = +A_{ex} \sum_{k=1}^3 (\vec{\nabla}(m_k))^2 \xrightarrow{\vec{\nabla} \bullet \vec{m} \rightarrow 0} A_{ex} |\vec{\nabla} \wedge \vec{m}|^2 \\ \xi_{ex} &\xrightarrow{dv \rightarrow 0} \iiint_V -A_{ex} (\vec{m} \bullet \Delta(\vec{m})) dV \\ &= \iiint_V +A_{ex} \sum_{k=1}^3 (\vec{\nabla}(m_k))^2 dV \\ &\xrightarrow{\vec{\nabla} \bullet \vec{m} \rightarrow 0} \iiint_V A_{ex} |\vec{\nabla} \wedge \vec{m}|^2 dV \end{aligned}$$

In case it is not reasonable to consider that $\vec{\nabla} \bullet \vec{m} \rightarrow 0$, then we still suggest defining a high order approximation of the magnetic Exchange energy density with $(A_{ex} |\vec{\nabla} \wedge \vec{m}|^2)$. In this case, it equals the usual 2nd order term $(-A_{ex} (\vec{m} \bullet \Delta(\vec{m})))$ added to $(-A_{ex} |\vec{\nabla} \bullet \vec{m}|^2)$, which shall balance the demagnetizing stray field energy due to $\vec{\nabla} \bullet \vec{m} \neq 0$. The boundary conditions stays fundamental to describe the demagnetizing stray field effects due to the surface magnetic poles.

APPENDIX 3: TMPT energy terms derivation

Averaging method or coarse-grained description of magnetization:

$$\vec{M}(x, t) = \langle \vec{m}(x, t) \rangle_x = \iiint w(x-x) \vec{m}(x, t) d^3x$$

It is a convolution over space with the kernel $w(x)$ being a real, non-negative and continuous function normalized to unity ($\iiint_V w(x) = 1$). It varies slowly over the size d_w of domains. The width of the kernel's plateau is larger than d_w and $w(x)$ tends towards 0 elsewhere ($w(x) \xrightarrow{x \gg d_w} 0$). Then we have

the following properties [27,29,30]:

$$\langle \vec{\nabla} \wedge \vec{m} \rangle = \vec{\nabla} \wedge \langle \vec{m} \rangle \quad \text{and} \quad \langle \vec{\nabla} \bullet \vec{m} \rangle = \vec{\nabla} \bullet \langle \vec{m} \rangle$$

Magnetic Exchange Energy:

$$\begin{aligned}
\mathcal{U}_{ex,i} &= J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \mathcal{U}_{ex} = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \langle u_{ex} \rangle = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \langle A_{ex} | (\vec{\nabla} \wedge \vec{m}) |^2 \rangle \\
&\Leftrightarrow \mathcal{U}_{ex,i} = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 A_{ex} \partial_{\varphi_i^2} | (\vec{\nabla} \wedge \langle \vec{m} \rangle) |^2 = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \frac{A_{ex}}{\mu_0^2} \partial_{\varphi_i^2} | (\vec{\nabla} \wedge ([V^2] \vec{\varphi})) |^2 \\
&\Leftrightarrow \mathcal{U}_{ex,i} = \left(\frac{\gamma_w}{K_{an}} \right)^4 (M_s^2 A_{ex}) \partial_{\varphi_i^2} \left(\sum_p (\varepsilon_{pqr} \partial_q V_{rs}^2) \varphi_s^2 \right) = \left(\frac{\gamma_w}{K_{an}} \right)^4 (M_s^2 A_{ex}) \left(\sum_p (\varepsilon_{pqr} \partial_q V_{rs}^2) \right) \\
&\Leftrightarrow \vec{\mathcal{U}}_{ex} = \left(\frac{\gamma_w}{K_{an}} \right)^4 (M_s^2 A_{ex}) ([\vec{\nabla}] \times [V^2]) \vec{\odot}^2 = C_{ex} \frac{\gamma_w^6}{K_{an}^6} ([\vec{\nabla}] \times [V^2]) \vec{\odot}^2 \text{ with } C_{ex} = \frac{A_{ex} K_{an}}{\gamma_w^2} M_s^2 = \frac{A_{ex} K_{an}}{\gamma_w^2} \frac{J_s^2}{\mu_0^2} [\text{n.u.}]
\end{aligned}$$

Magneto-Crystalline Anisotropy (case of uni-axial anisotropy):

$$\begin{aligned}
\mathcal{U}_{an,i} &= J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \mathcal{U}_{an} = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \langle u_{an} \rangle = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \left\langle \frac{k_{an}}{M_s^2} \left| \vec{m} \wedge \vec{u}_{an} \right|^2 \right\rangle \\
&\Leftrightarrow \mathcal{U}_{an,i} = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \frac{k_{an}}{M_s^2} \partial_{\varphi_i^2} \left| \left(\langle \vec{m} \rangle \wedge \vec{u}_{an} \right) \wedge \vec{u}_{an} \right|^2 = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \frac{k_{an}}{\mu_0^2 M_s^2} \partial_{\varphi_i^2} \left| \left(([V^2] \vec{\varphi}) \wedge \vec{u}_{an} \right) \wedge \vec{u}_{an} \right|^2 \\
&\Leftrightarrow \mathcal{U}_{an,i} = \frac{k_{an} \gamma_w^4}{K_{an}^4} \partial_{\varphi_i^2} \left| [V^2] \vec{\varphi} - \left(([V^2] \vec{\varphi}) \cdot \vec{u}_{an} \right) \vec{u}_{an} \right|^2 = \frac{k_{an} \gamma_w^4}{K_{an}^4} \partial_{\varphi_i^2} \sum_p \left(V_{pq}^2 \varphi_q - \frac{(V_{rs}^2 \varphi_s K_{an,r}) K_{an,p}}{K_{an}^2} \right)^2 \\
&\Leftrightarrow \mathcal{U}_{an,i} = \frac{k_{an} \gamma_w^4}{K_{an}^4} \left(V_{pi}^2 - \frac{(V_{rs}^2 K_{an,r}) K_{an,p}}{K_{an}^2} \right)^2 = \frac{k_{an} \gamma_w^4}{K_{an}^4} \left(\left([V^2] - \left(\frac{\vec{K}_{an} \cdot ([V^2] \cdot \vec{K}_{an})^T}{|\vec{K}_{an}|^2} \right)^T \right)^T \right)^2
\end{aligned}$$

$$\Leftrightarrow \vec{u}_{an} = C_{an} \frac{\gamma_w^4}{K_{an}^4} \left([V^2] - \left(\frac{\vec{K}_{an} \cdot ([V^2] \cdot \vec{K}_{an})^T}{|\vec{K}_{an}|^2} \right)^T \right)^T \text{ with } C_{an} = \frac{k_{an}}{K_{an}} [\text{n.u.}]$$

Magneto Striction Energy ($\chi = -3/2$ or $9/4$, $[L] = [\lambda]$ or $[\lambda^2]$ and $[T] = [s]$ or $[C]$):

$$\begin{aligned}
\mathcal{U}_{\lambda,i} &= J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \mathcal{U}_{\lambda} = J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \langle u_{\lambda} \rangle \\
&= J_s^2 \left(\frac{\gamma_w}{K_{an}} \right)^4 \partial_{\varphi_i^2} \left\langle \chi \frac{[\vec{m} \otimes \vec{m}]^{\circ}}{M_s^2} \left([L] \cdot [T] \cdot \frac{[\vec{m} \otimes \vec{m}]}{M_s^2} \right) \right\rangle \\
&\Leftrightarrow \mathcal{U}_{\lambda,i} = \left(\frac{\gamma_w}{K_{an}} \right)^4 \frac{\chi J_s^2}{\mu_0^2 M_s^2} \partial_{\varphi_i^2} \sum_p \sum_q \left(L_{pq} T_{pq} \left(\frac{\mu_0 m_p m_q}{M_s} \right)^2 \right) \\
&\quad \underset{2^{nd}}{\approx} \chi \left(\frac{\gamma_w}{K_{an}} \right)^4 \frac{J_s^2}{\mu_0^2 M_s^2} \partial_{\varphi_i^2} \sum_p \sum_q \left(L_{pq} T_{pq} (V_{pq}^2 \varphi_q)^2 \right) \\
&\Leftrightarrow \mathcal{U}_{\lambda,i} = \chi \left(\frac{\gamma_w}{K_{an}} \right)^4 \frac{J_s^2}{\mu_0^2 M_s^2} \partial_{\varphi_i^2} \sum_p \sum_q \left(V_{pq}^2 \varphi_q \cdot \left(L_{pq} T_{pq} V_{pq}^2 \varphi_q \right) \right) = \chi \left(\frac{\gamma_w}{K_{an}} \right)^4 \sum_p \left(V_{pi}^2 \cdot \left(L_{pi} T_{pi} V_{pi}^2 \right) \right) \\
&\Leftrightarrow \vec{u}_{\lambda} = \chi C_{\lambda} \frac{\gamma_w^4}{K_{an}^4} [V^2] \vec{\odot} \left(\frac{[L] \cdot [T] \cdot [V^2]}{K_{an}} \right) \text{ with } C_{\lambda} = 1 [\text{n.u.}]
\end{aligned}$$

APPENDIX 4: Magneto-Mechanical tensor properties

Magnetostriction (magneto-mechanical strain).

$$[\varepsilon] = \frac{3}{2} \begin{bmatrix} \lambda_{100} \left(\alpha_1^2 - \frac{1}{3} \right) & \lambda_{111} \alpha_1 \alpha_2 & \lambda_{111} \alpha_1 \alpha_3 \\ \lambda_{111} \alpha_2 \alpha_1 & \lambda_{100} \left(\alpha_2^2 - \frac{1}{3} \right) & \lambda_{111} \alpha_2 \alpha_3 \\ \lambda_{111} \alpha_3 \alpha_1 & \lambda_{111} \alpha_3 \alpha_2 & \lambda_{100} \left(\alpha_3^2 - \frac{1}{3} \right) \end{bmatrix} \text{ with } \alpha_1, \alpha_2, \alpha_3 \text{ the } \vec{m} \text{ cosine directors.}$$

Potential stress induced magnetostriction energy: $\mu_{\lambda s} = -\frac{3}{2} \frac{[\vec{m} \otimes \vec{m}]}{M_s^2} \circ \left([\lambda] :: [s] :: \frac{[\vec{m} \otimes \vec{m}]}{M_s^2} \right)$

with: $[\lambda] = \begin{bmatrix} \lambda_{100} & \lambda_{111} & \lambda_{111} \\ \lambda_{111} & \lambda_{100} & \lambda_{111} \\ \lambda_{111} & \lambda_{111} & \lambda_{100} \end{bmatrix}$ the tensor of magnetostriction coefficients.

Hooke law in rhombohedral crystals:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 = s_{23} \\ s_5 = s_{13} \\ s_6 = s_{12} \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}$$

Potential self magnetostriction energy: $\mu_{\lambda \varepsilon} = +\frac{9}{4} \frac{[\vec{m} \otimes \vec{m}]}{M_s^2} \circ \left([\lambda^2] :: [C] :: \frac{[\vec{m} \otimes \vec{m}]}{M_s^2} \right)$

with $[C] = \begin{bmatrix} C_{11} - C_{12} & C_{44} & C_{55} \\ C_{44} & C_{22} - C_{23} & C_{66} \\ C_{55} & C_{66} & C_{33} - C_{13} \end{bmatrix}$, the magneto-mechanical stiffness tensor.

with: $[\lambda^2] = \begin{bmatrix} \lambda_{100}^2 & \pm\lambda_{111}^2 & \pm\lambda_{111}^2 \\ \pm\lambda_{111}^2 & \lambda_{100}^2 & \pm\lambda_{111}^2 \\ \pm\lambda_{111}^2 & \pm\lambda_{111}^2 & \lambda_{100}^2 \end{bmatrix}$, the tensor of squared magnetostriction coefficients.

APPENDIX 5: Group of transformation that keeps invariant the Lagrangian and its action

We consider writing the entire space integral of the Lagrangian energy density with dissipation, *i.e.* the time derivation of the Action, in two different reference frames. This space integral of the Lagrangian energy density with dissipation named $\overrightarrow{\mathcal{L}}^{\#}$ and $\overrightarrow{\mathcal{R}}_{ed}^{\#}$ will be written $\overrightarrow{\mathfrak{S}}^{\#} = d\overrightarrow{\mathcal{L}}^{\#}/dt^{\#}$ in a reference frame $\mathfrak{N}^{\#} = (O^{\#}; x_1^{\#}, x_2^{\#}, x_3^{\#}, t^{\#})$ which is in Galilean motion at a speed \vec{v} as a function of another reference frame $\mathfrak{N} = (O; x_1, x_2, x_3, t)$ in which the space integral of the Lagrangian energy density with dissipation named $\overrightarrow{\mathcal{L}}$ and $\overrightarrow{\mathcal{R}}_{ed}$ will be written $\overrightarrow{\mathfrak{S}} = d\overrightarrow{\mathcal{L}}/dt$. The invariance principle is satisfied if and only if this integral does not depend on the reference frame in which it is calculated, *i.e.* if and only if $\overrightarrow{\mathfrak{S}}^{\#} = \overrightarrow{\mathfrak{S}}$ and $\overrightarrow{\mathcal{L}}^{\#} = \overrightarrow{\mathcal{L}}$. For clarity purpose of the proof, but without loss of generality, we consider that the speed \vec{v} is parallel to one first direction of space named $\overrightarrow{x_1}$, such that:

$$\overrightarrow{OO^{\#}}(t + dt) = \overrightarrow{OO^{\#}}(t) + \vec{v}dt = \overrightarrow{OO^{\#}}(t) + vdt\overrightarrow{x_1}$$

Considering that the two reference frames $\mathfrak{N}^{\#}$ and \mathfrak{N} are in uniform relative motion from each other along the $\overrightarrow{x_1}$ axis, we assume a linear transformation of the coordinates, to be determined:

$$\begin{cases} dx_1 = a_1 dx_1^{\#} + a_2 dt^{\#}, \\ dx_2 = dx_2^{\#}, \\ dx_3 = dx_3^{\#}, \\ dt = a_3 dx_1^{\#} + a_4 dt^{\#} \end{cases}$$

The invariance problem consists in finding the parameters (a_1, a_2, a_3, a_4) that ensure perforce $\overrightarrow{\mathfrak{S}}^{\#} = \overrightarrow{\mathfrak{S}}$:

$$\begin{aligned} \overrightarrow{\mathfrak{S}}^{\#} &= \frac{d\overrightarrow{\mathcal{L}}^{\#}}{dt^{\#}} = \iiint_{\infty} \left(\overrightarrow{\mathcal{L}}^{\#} - \overrightarrow{\mathcal{R}}_{ed}^{\#} \right) d^3 x^{\#} \\ &= -\iiint_{\infty} \left(\overrightarrow{\mathcal{U}}_{ex}^{\#} + \overrightarrow{\mathcal{U}}_{an}^{\#} + \overrightarrow{\mathcal{U}}_{\lambda s}^{\#} + \overrightarrow{\mathcal{U}}_{\lambda \varepsilon}^{\#} + \overrightarrow{\mathcal{U}}_{dm}^{\#} + \overrightarrow{\mathcal{R}}_{ed}^{\#} \right) d^3 x^{\#} \\ \overrightarrow{\mathfrak{S}} &= \frac{d\overrightarrow{\mathcal{L}}}{dt} = \iiint_{\infty} \left(\overrightarrow{\mathcal{L}} - \overrightarrow{\mathcal{R}}_{ed} \right) d^3 x = -\iiint_{\infty} \left(\overrightarrow{\mathcal{U}}_{ex} + \overrightarrow{\mathcal{U}}_{an} + \overrightarrow{\mathcal{U}}_{\lambda s} + \overrightarrow{\mathcal{U}}_{\lambda \varepsilon} + \overrightarrow{\mathcal{U}}_{dm} + \overrightarrow{\mathcal{R}}_{ed} \right) d^3 x \end{aligned}$$

Part of the energy densities without space or time operator are necessarily expressed with exactly the same formulation: $\overrightarrow{\mathcal{U}}_{an}^{\#} + \overrightarrow{\mathcal{U}}_{\lambda s}^{\#} + \overrightarrow{\mathcal{U}}_{\lambda \varepsilon}^{\#} + \overrightarrow{\mathcal{U}}_{dm}^{\#} = \overrightarrow{\mathcal{U}}_{an} + \overrightarrow{\mathcal{U}}_{\lambda s} + \overrightarrow{\mathcal{U}}_{\lambda \varepsilon} + \overrightarrow{\mathcal{U}}_{dm}$. Besides, we can write:

$$\begin{aligned}
 & -\iiint_{\infty} \left(\overrightarrow{\mathcal{R}}_{ex}^{\#} + \overrightarrow{\mathcal{R}}_{ed}^{\#} \right) d^3x^{\#} = -\iiint_{\infty} \left(C_{ex} \frac{\gamma_w^6}{K_{an}^5} ([\nabla^{\#}] \times [V^2])^{\overrightarrow{\odot}2} + \overrightarrow{\mathcal{R}}_{ed}^{\#} \right) d^3x^{\#} \\
 & = \left(-\iiint_{\infty} \left(C_{ex} \frac{\gamma_w^6}{K_{an}^5} (\nabla^{\#} \bullet ([V^2] \times ([\nabla^{\#}] \times [V^2])) + [V^2] \overrightarrow{\odot} ([\nabla^{\#}] \times ([\nabla^{\#}] \times [V^2])) \right) + \overrightarrow{\mathcal{R}}_{ed}^{\#} \right) d^3x^{\#} \\
 & \stackrel{\overrightarrow{\nabla}^{\#} \bullet [V^2] = \overrightarrow{0}}{=} C_{ex} \frac{\gamma_w^6}{K_{an}^5} \left(\iiint_{\infty} [V^2] \overrightarrow{\odot} ([\Delta^{\#}] [V^2]) d^3x^{\#} \right) - \iiint_{\infty} \left(\overrightarrow{\mathcal{R}}_{ed}^{\#} \right) d^3x^{\#} \text{ (with } [V^2] \overrightarrow{\rightarrow} \overrightarrow{0} \text{ outside the material)} \\
 & = -\iiint_{\infty} \left(C_{ex} \frac{\gamma_w^6}{K_{an}^5} \sum_p (\partial_{x_p}^{\#} [V^2])^{\overrightarrow{\odot}2} + \frac{1}{2} \frac{\gamma_w^4}{K_{an}^3} \tau \partial_{t^{\#}} [V^2] \overrightarrow{\odot} \tau \partial_{t^{\#}} [V^2] \right) d^3x^{\#}
 \end{aligned}$$

The relationships between the partial differential operators in $\mathfrak{R}^{\#}$ and the ones in \mathfrak{R} are as follow:

$$\begin{cases}
 \partial_{x_1}^{\#}(\cdot) = (a_1 \partial_{x_1}(\cdot) + a_3 \partial_t(\cdot)), \\
 \partial_{x_2}^{\#}(\cdot) = \partial_{x_2}(\cdot), \\
 \partial_{x_3}^{\#}(\cdot) = \partial_{x_3}(\cdot), \\
 \partial_{t^{\#}}(\cdot) = (a_2 \partial_{x_1}(\cdot) + a_4 \partial_t(\cdot))
 \end{cases}$$

Satisfying $\overrightarrow{\mathfrak{S}}^{\#} = \overrightarrow{\mathfrak{S}}$ will enforce the following conditions for (a_1, a_2, a_3, a_4) :

$$\begin{cases}
 \left(a_1^2 + \frac{a_2^2}{\vartheta^2} \right) = 1, \\
 \left(a_3^2 + \frac{a_4^2}{\vartheta^2} \right) = \frac{1}{\vartheta^2}, \\
 (a_1 a_3 + \frac{a_2 a_4}{\vartheta^2}) = 0
 \end{cases}$$

with $\vartheta = \sqrt{2C_{ex}} \frac{\gamma_w}{K_{an}^2} = \frac{M_s}{\tau} \sqrt{\frac{2A_{ex}}{K_{an}}}$ a characteristic speed of the magnetic structuring mechanisms.

The solution is:
$$\begin{cases}
 a_1 = a_4 = \alpha = \left(1 + \left(\frac{v}{\vartheta} \right)^2 \right)^{-1/2} = \sqrt{\frac{1}{1 + \beta^2}}, \\
 a_2 = \alpha v, \\
 a_3 = -\frac{\alpha v}{\vartheta^2} = -\frac{\alpha \beta^2}{v}
 \end{cases}$$
 with $\beta = \frac{v}{\vartheta}$

Which finally gives the proof of equations (45), (46) and the ones that follow.

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