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Ali Akhavi, Eda Cesaratto, Frédéric Paccaut, Pablo Rotondo, Brigitte Vallée

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# SHANNON WEIGHTS FOR BINARY DYNAMICAL RECURRENT SOURCES OF ZERO ENTROPY

ALI AKHAVI, EDA CESARATTO, FRÉDÉRIC PACCAUT, PABLO ROTONDO AND  
BRIGITTE VALLÉE

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ABSTRACT. A probabilistic source is defined as the set of infinite words (over a given denumerable alphabet) endowed with a probability  $\mu$ . The paper deals with general binary sources where the distribution of any symbol (0 or 1) may depend on an unbounded part of the previous history. The paper studies Shannon weights: whereas the classical Shannon entropy  $\mathcal{E}_\mu$  is the average amount of information brought by one symbol of the emitted word, the Shannon weight sequence deals with the average amount of information  $\underline{m}_\mu(n)$  that is brought by the emitted prefix of length  $n$ . For a source with a non zero entropy, the estimate  $\underline{m}_\mu(n) \sim \mathcal{E}_\mu \cdot n$  thus holds.

The paper considers the model of dynamical sources, where a source word is emitted as an encoded trajectory of a dynamical system of the unit interval, when endowed with probability  $\mu$ . It focus on sources with zero entropy and gives explicit constructions for sources whose Shannon weight sequence satisfies  $\underline{m}_\mu(n) = o(n)$ , with a prescribed behaviour. In this case, sources with zero entropy lead to dynamical systems built on maps with an indifferent fixed point. This class notably contains the celebrated Farey source, which presents well-known intermittency phenomena.

Methods are based on analytic combinatorics and generating functions, and they are enlarged, in the present dynamical case, with dynamical systems tools (mainly transfer operators).

## CONTENTS

1. Introduction	2
2. Sources, Shannon weights, block sources, waiting time	5
2.1. General sources	5
2.2. Shannon weights	6
2.3. Binary sources and number of ones.	7
2.4. Recurrence, waiting time and block sources.	8
2.5. Distribution of the waiting time and generating function $Q$ .	10
3. Dynamical sources of various types.	10
3.1. A class of dynamical systems.	10
3.2. Generating dynamical systems	12
3.3. Dynamical systems in the Class $\mathcal{DR}$ .	13
3.4. Recurrence.	14
3.5. Block source	15
3.6. The Good Class	16
3.7. The Class $\mathcal{DRBGC}$ .	16
3.8. An emblematic instance: the Farey system.	16
4. Statement of the four main results.	17
4.1. The first main result: invariant densities.	17
4.2. The second main result: renewal equations	18
4.3. Singularity analysis: Towards the third main result	19

4.4.	Sequences of interest	20
4.5.	The third main result	20
4.6.	Towards the fourth result.	21
5.	Generating operators and generating functions. The first main theorem.	22
5.1.	Generating operators	22
5.2.	Quasi-inverse $(I - \mathbb{L}_{s,t})^{-1}$ .	22
5.3.	The generating function $\Lambda_\mu$ and the secant $S[\phi]$	23
5.4.	Alternative expression for $\Lambda_\mu$	25
5.5.	Properties of the operators for a source of the Good Class	26
5.6.	Entropy of the block source.	27
5.7.	Invariant measure for a source in the Class $DRBGC$ .	28
6.	Dynamical analysis: Proof of the second main result.	29
6.1.	Perturbation of the transfer operator as $v \rightarrow 1^-$ .	29
6.2.	Behaviour of $1 - \lambda(v, 1)$ as $v \rightarrow 1^-$	30
6.3.	Study of the generating functions $N_\mu$ and $M_\mu$ as $v \rightarrow 1^-$ .	31
6.4.	End of the proof of the second main result.	33
6.5.	Two types of renewal equations	34
7.	Proof of the third main result.	35
7.1.	Main properties of slowly varying functions	36
7.2.	An auxilliary result.	36
7.3.	Proof of the third main result Theorem 23.	37
8.	The Class $DRIS$	38
8.1.	Notion of strong indifferent fixed point : the class $DRIS$ .	38
8.2.	About our Class $DRIS$	40
8.3.	First results on the waiting time in the class $DRIS$ .	40
8.4.	Proof of the inclusion $DRIS \subset DRBGC$ .	43
9.	The Class $DRIL$ .	44
9.1.	Definition of the Class $DRIL$ .	44
9.2.	Precise estimates of the wtd sequence	45
9.3.	Exhibiting a $DRIL$ source with a prescribed Shannon weight.	46
9.4.	Remarks on the final result	47
9.5.	Some instances of the $DRIL$ Class.	47
9.6.	Application to the Farey source.	48
9.7.	Return to Shannon weights	49
	References	50

## 1. INTRODUCTION

*Sources and weights.* The paper studies probabilistic models on infinite words. In information theory contexts, a source is a process defined on the set of infinite words over a given finite alphabet  $\Sigma$ , endowed with a probability  $\mu$ . It is a sequence of random variables  $X_1 \dots X_n \dots$  where  $X_n$  is the  $n$ -th emitted symbol.

The paper focuses on binary sources ( $\Sigma = \{0, 1\}$ ) and studies their Shannon weight defined in Section 2.2. The classical Shannon entropy  $\mathcal{E}_\mu$  –when it exists– is the average amount of information enclosed in one symbol of the emitted word, whereas the Shannon weight sequence deals with the average amount  $\underline{m}_\mu(n)$  of information enclosed in the emitted prefix of length  $n$ . We are interested in the asymptotic behaviour of the sequence  $n \mapsto \underline{m}_\mu(n)$  (as  $n \rightarrow \infty$ ).

When the sequence  $n \mapsto (1/n)\underline{m}_\mu(n)$  has a limit, it is equal to the entropy  $\mathcal{E}_\mu$ ; when  $\mathcal{E}_\mu$  is positive, Shannon weights satisfy the estimate  $\underline{m}_\mu(n) \sim \mathcal{E}_\mu \cdot n$ , and they are thus of linear order. But, for a source of zero entropy, the Shannon weights will

be a main object of interest: their behaviour provide an extra knowledge on the probabilistic properties of the source, and the paper answers the general question:

*Describe explicit sources with a prescribed Shannon weight  $\underline{m}_\mu(n) = o(n)$ .*

*Our models of sources.* The paper studies a general binary source (with alphabet  $\Sigma = \{0, 1\}$ ) which is furthermore *recurrent*<sup>1</sup>; it is first based on the fundamental decomposition of the set  $\{0, 1\}^*$  of finite binary words,

$$(1) \quad \{0, 1\}^* = \mathcal{U}^* \cdot \{0\}^*,$$

that deals with the set  $\mathcal{U} = \{0\}^*1$  of blocks, each block being a sequence of zeroes ended by a symbol one. We consider mainly recurrent binary sources with zero entropy, described along the decomposition (1), and aim at estimating their Shannon weights.

The two simplest model of sources – the memoryless sources which emit iid symbols, or the fixed length (aperiodic) Markov chains, when the distribution of any next symbol only depends on a bounded number of previous symbols – have a strictly positive entropy. We then focus here on another general model of sources, which may present more complex correlation phenomena, the model of dynamical sources. Such a dynamical source is associated with a measured dynamical system  $(\mathcal{I}, T, \mu)$  of the unit interval  $\mathcal{I}$ , and a word emitted by the source is here the encoded trajectory of an input  $x \in \mathcal{I}$  under the probability  $\mu$  defined on  $\mathcal{I}$ . As we are mainly interested in sources of zero entropy, we are led to dynamical systems with indifferent fixed points, whose encoded trajectories present intermittency phenomena, with long runs of zeroes. A typical instance is the Farey source.

*Block source.* Starting with a binary recurrent source  $\mathcal{S}$ , the decomposition (1) naturally introduces another source –no longer a binary source– that indeed generates blocks and is called the block source, denoted as  $\mathcal{B}$ . The block source  $\mathcal{B}$  will be a central object in our analysis. We deal within a general context where the source  $\mathcal{S}$  is a dynamical source whose block source  $\mathcal{B}$  has nice properties : it is associated with a dynamical system of the *Good Class*, and it has notably an invariant probability measure  $\nu$  with a finite positive entropy  $\mathcal{E}_\nu(\mathcal{B})$ . In this context, the block source  $\mathcal{B}$  is often more “regular” and easier to study than the initial source  $\mathcal{S}$ , and we often “replace”<sup>2</sup> in the study the source  $\mathcal{S}$  by the source  $\mathcal{B}$ .

*Generating functions.* We extensively use methods of analytic combinatorics, in the same vein as in the book of Flajolet and Sedgewick [FS09]. We first describe a source  $\mathcal{S}$  with probability  $\mu$  via the probabilities  $p(w) = p_\mu(w)$  that the emitted (infinite) word begins with the prefix  $w$ . Due to (1), the number  $n(w)$  of ones in the prefix  $w$  is also central, and we associate with the source  $\mathcal{S}$  a trivariate generating function

$$(2) \quad \Lambda_\mu(v, t, s) := \sum_{w \in \Sigma^*} v^{|w|} t^{n(w)} p(w)^s.$$

The mean number  $\underline{n}_\mu(n)$  of ones and the mean average amount of information  $\underline{m}_\mu(n)$  contained in a prefix of length  $n$  are our final objects of interest, but the distribution  $q_\mu(n) = \mu[W > n]$  of the waiting time  $W$  of the symbol one is also an important (auxilliary) parameter. There are finally three generating functions (gf’s in shorthand notation) of interest: the gf’s  $N_\mu(v)$  of the number of ones,  $M_\mu(v)$  of

<sup>1</sup>Such a source produces infinite words which contain almost surely an infinite number of ones. A precise definition is given in Definition 3.

<sup>2</sup>As a main instance of this fact, the Farey source is “replaced” by the source defined by the Gauss map and associated with continued fraction expansions

Shannon weights and  $Q_\mu(v)$  of the waiting time,

$$(3) \quad Q_\mu(v) = \sum q_\mu(n) v^n, \quad N_\mu(v) = \sum \underline{n}_\mu(n) v^n, \quad M_\mu(v) = \sum \underline{m}_\mu(n) v^n.$$

Remark that the gf's  $N_\mu(v)$  and  $M_\mu(v)$  are (formal) derivatives of  $\Lambda_\mu(v, t, s)$  with respect to  $t$  or to  $s$  at  $(t, s) = (1, 1)$ .

*Dynamical analysis.* All our analyses, performed in the dynamical context, are based on the *dynamical analysis method* designed by Vallée and then extended by Cesaratto and Vallée in various papers (see for instance [BV05], [V01], [CV15]). This method mixes analytic combinatorics tools and dynamical systems tools, in a general framework where the (secant) transfer operator of the dynamical system is used as a generating operator which itself generates the fundamental probabilities  $p(w)$  of the source. This method was already widely used in the analysis of Euclidean algorithms ([BV05], [V03],[V98]), in arithmetical studies ([CV06]) or in contexts of information theory (see [V01]), but it occurs for the first time in the study of Shannon weights.

Using decomposition (1), and directly dealing with generating functions associated with dynamical sources, the paper first obtains in Theorem 26 a precise (formal) expression of the trivariate generating function  $\Lambda_\mu(v, t, s)$  in terms of transfer operators, where the weighted transfer operator  $\mathbb{G}_{v,s}$  of the block system  $\mathcal{B}$  plays a central role. We are then led to dynamical systems  $\mathcal{S}$  for which the block system  $\mathcal{B}$  belongs to an important subclass of dynamical systems, the *Good Class*, already mentioned and deeply studied in [CV15]. There, the operator  $\mathbb{G}_{v,s}$  fullfills (on a convenient functional space) nice dominant spectral properties, and it will be possible to take the derivatives of function  $\Lambda_\mu$ .

*The four main results.* The paper studies, as its first main result stated in Theorem 19, invariant densities for the initial source  $\mathcal{S}$  and for the block source  $\mathcal{B}$ . Then, it obtains in Theorem 20, two relations (and one of renewal type) between the three gf's  $Q_\mu(v)$ ,  $N_\mu(v)$  and  $M_\mu(v)$ . It remains to “extract” the coefficients  $\underline{n}_\mu(n)$ ,  $\underline{m}_\mu(n)$ ,  $q_\mu(n)$  of the gf's. As these gf's have positive coefficients, a good knowledge of their behaviour on the real segment  $[0, 1[$  (as  $v \rightarrow 1^-$ ) provides, with Abelian and Tauberian Theorems, asymptotic estimates for the mean number of ones  $\underline{n}_\mu(n)$  and Shannon weights  $\underline{m}_\mu(n)$  in terms of  $q_\mu(n)$ , described in our third main result, Theorem 23. It derives in particular an (indirect) relation between the mean number  $\underline{n}_\mu(n)$  of ones and the Shannon weights, namely

$$\underline{m}_\mu(n) \sim \mathcal{E}_\nu(\mathcal{B}) \underline{n}_\mu(n) \quad (n \rightarrow \infty),$$

that involves the entropy  $\mathcal{E}_\nu(\mathcal{B})$  of the block system (wrt to its invariant measure  $\nu$ ). The previous relation can be viewed as an extension of the Abramov formula, – which relates the the entropy  $\mathcal{E}_\pi(\mathcal{S})$  of the initial source with respect to its invariant measure  $\pi$  and the entropy  $\mathcal{E}_\nu(\mathcal{B})$  of the block source, when they are both positive and finite– to the case when the initial dynamical system  $\mathcal{S}$  has zero entropy and possesses a block system in the Good Class.

The fourth main result, whose main steps are summarized in Section 4.6, is described in Theorem 49: it explains how to find an explicit source with given Shannon weights and answers the central question of the paper.

*Related works.* In the context of dynamical systems, the number of ones has been already deeply studied, notably by Thaler and Aaronson, with methods of Infinite Ergodic Theory ([A81] [T83]). Aaronson provides a renewal equation of the same type as ours, but with respect to another type of measure, and only for the number of ones. These authors do not study Shannon weights. We compare our results with methods of Infinite Ergodic Theory in Section 6.

Our results are obtained via analytic combinatorics and dynamical analysis. Our approach is furthermore mainly applied to sources with zero entropy; even though this zero-entropy class is natural and has been remarked a long time ago, it is less understood and less investigated. There is yet a recent renewal of interest for these sources: the paper [APV22] describes a general process for building in a natural way sources of zero entropy from sources of positive entropy; moreover, the paper [BCRS23] describes and studies new extensions of entropies of various kinds to various weights.

*Plan of the paper.* Section 2 introduces the main objects (sources, Shannon weights, waiting time, block source), and Section 3 first transfers the main concepts of Section 2 to the dynamical context. Section 4 states the four main results. Then Section 5 describes the main tools –the trivariate gf  $\Lambda$  and the three gf's  $Q, N, M$ – and proves the first theorem on invariant densities (Theorem 19). The next Section 6 proves the second main result (Theorem 20). Then, Section 7 describes the precise version of Abelian and Tauberian Theorems of use (that involves slowly varying functions) and explains how to deduce the third result (Theorem 23) from Theorem 20. The sequel of the paper (with Sections 8 and 9) builds explicit instances, based on maps with indifferent fixed points, and leads to the answer of the main question of the paper.

## 2. SOURCES, SHANNON WEIGHTS, BLOCK SOURCES, WAITING TIME

The present section describes the main objects of interest: sources in Section 2.1, with various points of view, then the main costs of interest: Shannon weights in Section 2.2, and number of ones in Section 2.3. With the waiting time  $W$  of symbol one, the word decomposes into a sequence of blocks, and Section 2.4 introduces the recurrence property (Property  $\mathcal{R}$ ), that allows to define the source which emits blocks, called the block source. This will be a central object of the paper.

**2.1. General sources.** A general source is a probabilistic process, denoted by  $\mathcal{S}$ , that emits infinite words with symbols from a given alphabet  $\Sigma$  (finite or denumerable), one symbol at each discrete time  $t = 1, \dots, t = n, \dots$ . Here  $\mathbb{N}$  will denote the set  $\{1, 2, \dots\}$  of positive integers.

*Sequence of random variables.* Consider some abstract probability space  $(\Omega, \mathcal{O}, \text{Pr})$ . When, for  $n \geq 1$ ,  $X_n$  (defined on  $\Omega$ ) denotes the random variable whose value at  $\omega \in \Omega$  is the symbol  $\alpha_n$  emitted at time  $t = n$ , namely  $\alpha_n = X_n(\omega)$ , the source  $\mathcal{S}$  will be identified with the sequence  $X = (X_n)_{n \geq 1}$  of random variables.

In the sequel, any sequence (finite or infinite) of symbols will be identified with the concatenation of its terms (written without commas). Any word emitted by the source is thus a one-sided infinite word  $\alpha = X(\omega) = \alpha_1 \dots \alpha_n \dots \in \Sigma^{\mathbb{N}}$ .

*Fundamental domains, fundamental probabilities and Kolmogorov Property.* For each  $k \geq 1$ , the projection  $\pi_k : \Sigma^{\mathbb{N}} \rightarrow \Sigma^k$  associates with the infinite word  $\alpha \in \Sigma^{\mathbb{N}}$  its finite prefix  $w = (\alpha_1 \alpha_2 \dots \alpha_k)$ , and the set

$$(4) \quad \mathcal{I}_w = \pi_{|w|}^{-1}(\{w\}) \text{ gathers the elements of } \Sigma^{\mathbb{N}} \text{ that begin with the prefix } w.$$

It is called the cylinder (or the fundamental domain) associated with the prefix  $w$ . The space  $\Sigma^{\mathbb{N}}$  is endowed with the product  $\sigma$ -algebra, i.e., the  $\sigma$ -algebra  $\mathcal{F}$  generated by the sets  $\{\pi_k^{-1}(w), w \in \Sigma^k, k \geq 1\}$ . Since these sets are a base for the product topology,  $\mathcal{F}$  is also the Borel  $\sigma$ -algebra on  $\Sigma^{\mathbb{N}}$ . Then, the sequence  $\mathcal{P}$  of fundamental partitions  $\mathcal{P}_k$  of order  $k$ ,

$$(5) \quad \mathcal{P} = (\mathcal{P}_k), \quad \mathcal{P}_k = \{\mathcal{I}_w \mid w \in \Sigma^k\}, \quad p(w) = \mu(\mathcal{I}_w),$$

generates the  $\sigma$ -algebra  $\mathcal{F}$ . The distribution  $\mu = \mu_X$  of the sequence of random variables  $X = (X_n)_{n \geq 1}$ , is defined on  $\mathcal{F}$ , and is thus completely specified by the fundamental probabilities  $p(w) = \mu(\mathcal{I}_w)$  that a word begins with the prefix  $w$ : this is known as the Kolmogorov principle.

*A recursive point of view on the source.* We consider the following two mappings  $\sigma$  and  $T$  –  $T$  is also called the shift – defined on  $\Sigma^{\mathbb{N}}$ : For an infinite word  $\alpha \in \Sigma^{\mathbb{N}}$ , with  $\alpha = \alpha_1\alpha_2, \dots$ ,  $\sigma(\alpha) = \alpha_1$  is the first symbol of  $\alpha$  and  $T(\alpha)$  is the infinite suffix  $(\alpha_2\alpha_3 \dots)$ , so that the symbol  $\alpha_n$  emitted at time  $n$  is  $\alpha_n = \sigma(T^{n-1}(\alpha))$  and the (infinite) emitted word  $\alpha$  is then written as

$$(6) \quad \alpha = (\sigma(\alpha), T(\alpha)) = (\sigma(\alpha), \sigma(T(\alpha)), \dots, \sigma(T^n(\alpha)) \dots).$$

*Costs.* The main objects of our study are *costs*. A cost  $c$  is a map  $c : \Sigma^* \rightarrow \mathbb{R}$ , and, for any  $k \geq 0$ , we consider the mean value  $\underline{c}_\mu(k)$  of  $c$  on  $\Sigma^k$ ,

$$(7) \quad \underline{c}_\mu(k) := \sum_{w \in \Sigma^k} p_\mu(w) c(w).$$

We wish to study the asymptotic behaviour of the sequence  $(\underline{c}_\mu(k))_{k \geq 0}$ . As it is usual in analytic combinatorics, we deal with the associated generating function

$$(8) \quad C_\mu(v) = \sum_{k \geq 0} \underline{c}_\mu(k) v^k.$$

The cost  $c$  is increasing if

$$c(w \cdot i) \geq c(w) \text{ for any } w \in \Sigma^* \text{ and } i \in \Sigma,$$

where  $w \cdot i$  denotes the concatenation of the word  $w$  and the letter  $i$ . This means that the sequence  $k \mapsto c \circ \pi_k$  is increasing and the cost  $c$  can be defined on the infinite words as a cost  $c : \Sigma^{\mathbb{N}} \rightarrow \mathbb{R} \cup \{+\infty\}$  with

$$(9) \quad c(\alpha) := \lim_{k \rightarrow \infty} c \circ \pi_k(\alpha) = \sup\{c(w) \mid w \text{ finite prefix of } \alpha\}.$$

Then the following holds

$$\underline{c}_\mu(k) = \mathbb{E}_\mu[c \circ \pi_k], \quad \underline{c}_\mu(k) \geq \underline{c}_\mu(k-1),$$

and the generating function  $C_\mu(v)$  defined in (8) satisfies the following:

**Lemma 1.** *For any increasing cost  $c$ , the generating function  $(1-v)C_\mu(v)$  has positive coefficients and the partial sums of these coefficients coincide with  $\underline{c}_\mu(n)$ .*

*Proof.* The series  $(1-v)C_\mu(v)$  satisfies

$$(1-v)C_\mu(v) = \sum_{k \geq 0} \underline{c}_\mu(k) v^k - \sum_{k \geq 0} \underline{c}_\mu(k) v^{k+1} = \underline{c}_\mu(0) + \sum_{k \geq 1} (\underline{c}_\mu(k) - \underline{c}_\mu(k-1)) v^k.$$

□

**2.2. Shannon weights.** We consider the cost  $m$  defined on  $\Sigma^*$  that associates with the prefix  $w \in \Sigma^*$  the amount of information it brings. Then the cost  $m$  satisfies<sup>3</sup>

$$m(w) = |\log p_\mu(w)| \text{ if the prefix } w \text{ is emitted,} \quad m(w) = 0 \text{ if not,}$$

and is increasing. As in (7) and (8), we associate,

$$\underline{m}_\mu(k) = \sum_{w \in \Sigma^k} p_\mu(w) m(w), \quad M_\mu(v) = \sum_{k \geq 0} \underline{m}_\mu(k) v^k.$$

<sup>3</sup>We may also consider that any prefix  $w$  (even though it is not emitted) always provides the supplementary information relative to its length  $|w|$  and, in this case, the amount of information brought by the prefix  $w$  is

$$\tilde{m}(w) = \log |w| + m(w).$$

The mean value  $\underline{m}_\mu(k)$  is written as<sup>4</sup>

$$(10) \quad \underline{m}_\mu(k) = \sum_{w \in \Sigma^k} \llbracket p_\mu(w) \neq 0 \rrbracket p_\mu(w) |\log p_\mu(w)| = \sum_{w \in \Sigma^k} p_\mu(w) |\log p_\mu(w)|,$$

the second equality being due to the convention  $x \log x = 0$  when  $x = 0$ . Thus  $\underline{m}_\mu(k)$  coincides with the joint entropy  $H(X_1, \dots, X_k)$  of the random variables that define the source (see [CT06]).

**Definition 2.** For the source  $(\Sigma^{\mathbb{N}}, \mu)$ ,

- the sequence  $k \mapsto \underline{m}_\mu(k)$  is called the sequence of Shannon weights;
- the limit of the sequence  $(1/k) \underline{m}_\mu(k)$  –when it exists– is called the entropy of the source, denoted as  $\mathcal{E}_\mu$ .

In information theory contexts (see for instance [CT06]), the entropy of the source is sometimes called *entropy rate*. A sufficient condition for the convergence of the sequence  $(1/k) \underline{m}_\mu(k)$  is the sub-additivity of the sequence  $k \mapsto \underline{m}_\mu(k)$ . And a sufficient condition for this sub-additivity is the stationarity of the sequence  $(X_n)_{n \in \mathbb{N}}$  or equivalently (See [PY98]) the fact that  $\mu$  is invariant by the shift  $T$ , i. e.,

$$\mu(T^{-1}G) = \mu(G) \text{ for any } G \in \mathcal{F}.$$

When the entropy  $\mathcal{E}_\mu$  is positive, the Shannon weights  $\underline{m}_\mu(k)$  are of linear order. But, in the case of zero entropy, the Shannon weights are  $o(n)$  and their precise asymptotic study provides an extra interesting knowledge on the source.

The idea of extending entropy into a sequence of weights is not new and has already been proposed (for instance) in the paper [BCRS23], notably in the context of dynamical sources (see Section 9.7). However, the weights introduced there extend other types of entropy, defined in Dajani and Fieldsteel [DF01], (not the Shannon entropy), and are described in Definitions 2.1 and 2.2 of [BCRS23]. To the best of our knowledge, the interest in the precise asymptotic study of Shannon weights, that describes the evolution of  $|\log p(w)|$  when  $|w|$  tends to  $\infty$ , appears to be new. With the point of view of Footnote 3, the average amount of information on  $\Sigma^k$  is

$$\tilde{m}_\mu(k) = \log k + \underline{m}_\mu(k),$$

and this version of Shannon weights satisfies  $\tilde{m}_\mu(k) \geq \log k$ .

**2.3. Binary sources and number of ones.** The paper is devoted to the case of binary sources  $\mathcal{S}$ , i.e., sources with a binary alphabet  $\Sigma = \{0, 1\}$ . There is a fundamental decomposition

$$(11) \quad \{0, 1\}^* = \mathcal{U}^* \cdot \mathcal{Z},$$

$$(12) \quad \text{with } \mathcal{U} := \{0\}^* 1 = \{0^{u-1} 1 \mid u \geq 1\}, \quad \mathcal{Z} := \{0\}^* = \{0^u \mid u \geq 0\}.$$

An element  $U$  of  $\mathcal{U}$  is called a block, and an element of  $\mathcal{Z}$  is called a zero-tail. The set  $\mathcal{U}$  is a set of words and may thus be called a code (in coding theory). This set  $\mathcal{U}$  has an important property: a proper prefix of a codeword  $U$  of  $\mathcal{U}$  does not belong to  $\mathcal{U}$ , and  $\mathcal{U}$  is called a prefix code (in coding contexts).

The fundamental decomposition (12) provides, for any finite word  $w$ , the writing

$$(13) \quad \left\{ \begin{array}{l} w = (0^{u_1-1} 1) \dots (0^{u_{\ell-1}-1} 1) (0^{u_\ell-1} 1) \quad 0^{u_0} \quad (\ell \geq 1) \\ w = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0^{u_0}, \quad (\ell = 0 \text{ or } w = \epsilon) \end{array} \right\},$$

where  $\ell \geq 0$  is the number of ones contained in  $w$ , the integers  $u_i$  satisfy  $u_i \geq 1$  (for  $1 \leq i \leq \ell$ ) and  $u_0 \geq 0$ .

<sup>4</sup>For an event  $A$ ,  $\llbracket A \rrbracket$  is the Iverson's bracket equal to 1 if  $A$  arises and 0 if not

*Number of ones.* We are thus led to another important cost  $n : \Sigma^* \rightarrow \mathbb{N}$ , where  $n(w)$  equals the number of ones in  $w$  (it coincides with the number of blocks). The cost  $n$  is increasing, and, as in (7) and (8), we associate

$$\underline{n}_\mu(k) = \sum_{w \in \Sigma^k} p_\mu(w) n(w), \quad N_\mu(v) = \sum_{k \geq 0} \underline{n}_\mu(k) v^k.$$

The present paper studies the asymptotic behaviour of the sequence  $k \mapsto \underline{n}_\mu(k)$ . As the cost  $n$  is increasing, it may be defined on infinite words  $\alpha$ , with values in  $\mathbb{N} = \mathbb{N} \cup \{+\infty\}$ , as in (9) and the equality holds

$$\underline{n}_\mu(k) = \mathbb{E}_\mu[n \circ \pi_k].$$

The random variable  $n \circ \pi_k$  is expressed via an ergodic sum in terms of the set  $\mathcal{J}$  of words which begin with the symbol one,

$$(14) \quad n \circ \pi_k(\alpha) = \sum_{\ell=0}^{k-1} \mathbf{1}_{\mathcal{J}} \circ T^\ell(\alpha), \quad \mathcal{J} := \{\alpha \in \Sigma^{\mathbb{N}} \mid \sigma(\alpha) = 1\},$$

and this is why (Infinite) Ergodic Theory arises in a natural way in the study of  $\underline{n}_\mu(k)$ . The present paper uses another direction, and presents alternative proofs for the asymptotic estimates of the mean number  $\underline{n}_\mu(k)$  of ones, based on analytic combinatorics, dynamical methods, and functional analysis. We think that this method of proof may be of independent interest.

**2.4. Recurrence, waiting time and block sources.** We consider the decreasing sequence  $\mathcal{N}_\ell$  of subsets of  $\{0, 1\}^{\mathbb{N}}$ , and its intersection  $\mathcal{N}$ ,

$$(15) \quad \mathcal{N}_\ell := \{\alpha \in \{0, 1\}^{\mathbb{N}} \mid n(\alpha) \geq \ell\} \quad (\ell \geq 1), \quad \mathcal{N} := \bigcap_{\ell \geq 1} \mathcal{N}_\ell,$$

so that  $\mathcal{N}$  is the set of (infinite) words having an infinite number of ones. Then, for any  $\alpha \in \mathcal{N}$ , and for any  $k_0$ , there exists  $k \geq k_0$  for which  $T^k(\alpha) \in \mathcal{J}$ .

*Recurrence.* We consider sources that satisfy, when restricted to the set  $\mathcal{N}$ , a recurrence property with respect to the occurrence of symbol one, that is expressed in terms of the fundamental domains  $\mathcal{I}_w$  introduced in (4):

**Definition 3.** *The source  $(\Sigma^{\mathbb{N}}, \mu)$  is recurrent if*

*for any  $w \in \Sigma^*$ , the sequence  $\mu(\mathcal{I}_{w \cdot 0^n})$  tends to 0 as  $n \rightarrow \infty$ .*

Remark that, if the source  $(\Sigma^{\mathbb{N}}, \mu)$  is recurrent, then any source  $(\Sigma^{\mathbb{N}}, \mu')$  with  $\mu \equiv \mu'$ <sup>5</sup> is recurrent too.

**Proposition 4.** [Recurrence]

(a) (i) *When the source  $(\{0, 1\}^{\mathbb{N}}, \mu)$  is recurrent, one has, with  $p = p_\mu$*

$$p(w) = \sum_{U \in \mathcal{U}} p(w \cdot U) \quad \text{for any } w \in \Sigma^*.$$

(ii) *For any  $\ell \geq 1$ , for any sequence  $(U_1, U_2, \dots, U_\ell)$  of  $\ell$  blocks,*

$$p(U_1 \cdot U_2 \cdot \dots \cdot U_\ell) = \sum_{U \in \mathcal{U}} p(U_1 \cdot U_2 \cdot \dots \cdot U_\ell \cdot U);$$

$$\sum_{(U_1, U_2, \dots, U_\ell) \in \mathcal{U}^\ell} p(U_1 \cdot U_2 \cdot \dots \cdot U_\ell) = 1;$$

(b) *The equivalence holds, and involves the set  $\mathcal{N}$  defined in (15),*

$$\text{The source } (\{0, 1\}^{\mathbb{N}}, \mu) \text{ is recurrent} \iff \mu(\mathcal{N}) = 1.$$

*Proof.* Item (a)(i). For any  $w \in \Sigma^*$ , one has

$$p(w) = p(w \cdot 0) + p(w \cdot 1) = p(w \cdot 00) + p(w \cdot 01) + p(w \cdot 1)$$

so that, for any  $i$ ,

$$p(w) = p(w \cdot 0^i) + p(w \cdot 0^{i-1}1) + p(w \cdot 0^{i-2}1) + p(w \cdot 0^{i-3}1) + \dots + p(w \cdot 1).$$

<sup>5</sup>Two measures are equivalent if they admit the same subsets of zero measure

Letting  $i \rightarrow \infty$  proves Item (a)(i).

Item (a)(ii). We apply (a)(i) to  $w = U_1 \cdot U_2 \cdot \dots \cdot U_\ell$ .

Item (a)(iii) is proven by induction. For  $\ell = 1$ , this is (a)(i) applied to  $w = \epsilon$ . Now

$$\begin{aligned} \sum_{(U_1, U_2, \dots, U_\ell, U_{\ell+1}) \in \mathcal{U}^{\ell+1}} p(U_1 \cdot U_2 \cdot \dots \cdot U_\ell \cdot U_{\ell+1}) &= \sum_{U \in \mathcal{U}} \sum_{(U_1, U_2, \dots, U_\ell) \in \mathcal{U}^\ell} p(U_1 \cdot U_2 \cdot \dots \cdot U_\ell \cdot U) \\ &= \sum_{(U_1, U_2, \dots, U_\ell) \in \mathcal{U}^\ell} \sum_{U \in \mathcal{U}} p(U_1 \cdot U_2 \cdot \dots \cdot U_\ell \cdot U) \end{aligned}$$

and we use (a)(ii) together the induction hypothesis. .

Item (b). The set  $\mathcal{N}_\ell$  coincides with the set of infinite words  $X$  which begin with a sequence of  $\ell$  blocks. Then

$$\mu[\mathcal{N}_\ell] = \sum_{(U_1, U_2, \dots, U_\ell) \in \mathcal{U}^\ell} p(U_1 \cdot U_2 \cdot \dots \cdot U_\ell)$$

that equals 1 with Item (a)(ii). Conversely, if a source is not recurrent, then there exists a finite word  $w$  such that  $\lim_i p(w0^i)$  is strictly positive. It gives rise to an infinite word  $\alpha = w \cdot 0^\mathbb{N}$  for which  $\mu(\alpha)$  is strictly positive and  $n(\alpha) = n(w)$  is finite. Then  $\alpha$  does not belong to  $\mathcal{N}$  and  $\mu[\mathcal{N}]$  is strictly less than 1.  $\square$

*Waiting time.* With the set  $\mathcal{J}$  of words which begin with the symbol one, described in (14), the *waiting time*  $W$  to hit  $\mathcal{J}$  is defined on the set  $\mathcal{N}$  as

$$W(\alpha) = 1 + \min\{k \mid \sigma(T^k(\alpha)) = 1\} = 1 + \min\{k \geq 0 \mid T^k(\alpha) \in \mathcal{J}\}.$$

There is (1+) in the formula because the symbol produced at time  $t$  is the symbol  $\sigma(T^{t-1}(\alpha))$ . This waiting time  $W$  is the first time  $t \geq 1$  when there is a symbol ‘1’ at position  $t$ . This gives rise to three random variables, all defined on  $\mathcal{N}$ , the first block  $U$ , its length  $u$ , and the block shift  $\widehat{T}$

$$U(\alpha) = 0^{W(\alpha)-1}1, \quad u = W(\alpha), \quad \widehat{T}(\alpha) = T^{W(\alpha)}(\alpha).$$

The block shift  $\widehat{T}$  is also called the block shift because it “jumps” over the block. The infinite word  $\alpha \in \mathcal{N}$  is thus written as

$$\alpha = (W(\alpha), \widehat{T}(\alpha)),$$

and this defines, as in (6), and on the set  $\mathcal{N}$ , a source  $\mathcal{B}$  with alphabet  $\mathbb{N}$ , with

$$(16) \quad \sigma_{\mathcal{B}}(\alpha) = W(\alpha), \quad T_{\mathcal{B}}(\alpha) = \widehat{T}(\alpha).$$

*Block source.* When the source  $\mathcal{S}$  is recurrent, the map  $\widehat{T} : \mathcal{N} \rightarrow \mathcal{N}$  is defined  $\mu$ -almost everywhere, and any infinite word of  $\mathcal{N}$  emitted by  $\mathcal{S}$  is encoded as in (16). This defines on  $\mathcal{N}$  a source  $\mathcal{B}$  with alphabet  $\mathbb{N}$ , and, by definition of this source  $\mathcal{B}$ , the equality holds for  $U_i = 0^{u_i-1}1$

$$\mu[\mathcal{S} \text{ emits the prefixes } U_1, U_2, \dots, U_\ell] = \widehat{\mu}[\mathcal{B} \text{ emits the prefix } u_1, u_2, \dots, u_\ell].$$

This leads to the next definition which introduces one of the main tools in the present paper.

**Definition 5.** [Block source] *When the source  $\mathcal{S} = (\{0, 1\}^\mathbb{N}, \mu)$  is recurrent, it gives rise on the set  $\mathcal{N}$  with full measure  $\mu[\mathcal{N}] = 1$  to a block source  $\mathcal{B} = (\mathbb{N}^\mathbb{N}, \widehat{\mu})$ , where  $\widehat{\mu}$  is defined by the equalities*

$$p_{\widehat{\mu}}(u_1 u_2 \dots u_\ell) = p_\mu(0^{u_1-1}1 \cdot 0^{u_2-1}1 \dots 0^{u_\ell-1}1),$$

*that relate the fundamental probabilities  $p_\mu$  of the source  $\mathcal{S}$  and the fundamental probabilities  $p_{\widehat{\mu}}$  of the source  $\mathcal{B}$ .*

**2.5. Distribution of the waiting time and generating function  $Q$ .** For a recurrent source  $\mathcal{S} = (\{0, 1\}^{\mathbb{N}}, \mu)$ , and with Proposition 4, the waiting time  $W$  and the block source  $\mathcal{B}$  are well-defined (almost everywhere), and one thus deals with three main objects:

*Level sequences related to the distribution of  $W$  (wtd sequences).* There are two level sets sequences attached with the waiting time  $W$ , namely,

$$n \mapsto [W > n], \quad n \mapsto [W = n].$$

The probabilities of the associated events define the waiting time distribution (in shorthand *wtd*),

$$(17) \quad \left\{ \begin{array}{l} q_{\mu}(k) = \mu[W > k] = \mu(\mathcal{I}_{0^k}) \quad (k \geq 0) \\ r_{\mu}(k) = \mu[W = k] = \mu(\mathcal{I}_{0^{k-1}1}) \quad (k \geq 1) \end{array} \right\},$$

and, for a recurrent source, the sequence  $q_{\mu}(k)$  tends to 0.

*Generating function  $Q_{\mu}(v)$ .* The generating function  $Q_{\mu}(v)$  of the sequence  $k \mapsto q_{\mu}(k)$  is defined as

$$(18) \quad Q_{\mu}(v) = \sum_{k \geq 0} q_{\mu}(k) v^k \quad \text{and satisfies} \quad Q_{\mu}(v) \rightarrow \mathbb{E}_{\mu}[W], \quad (v \rightarrow 1^-).$$

### 3. DYNAMICAL SOURCES OF VARIOUS TYPES.

We now consider particular sources of the previous Section, created by measured dynamical systems; we first define this concept in Section 3.1, based on the notion introduced in [V01], and we introduce in Section 3.2 the notion of generating dynamical sources, associated with measured dynamical systems. The general framework of Section 2.4 is then transferred in Section 3.3 into the framework of dynamical sources. We define in particular the Class  $\mathcal{DR}$  which gives rise to recurrent sources, then, the Class  $\mathcal{DRBG}$  which gathers sources of the Class  $\mathcal{DR}$  for which the block source is of Good Class. This Class will be central in dynamical analysis of Sections 5 and 6; it contains an emblematic source, the Farey source described in Section 3.8.

We use the same notations as in Section 2. The alphabet (finite or denumerable) is denoted by  $\Sigma$ , an infinite word (i.e. an element of  $\Sigma^{\mathbb{N}}$ ) is denoted by  $\alpha$  whereas a finite word (i.e. an element of  $\Sigma^*$ ) is denoted by  $w$ . Moreover, the Lebesgue measure on the unit interval  $\mathcal{I} = [0, 1]$  is denoted by  $\tau$ .

#### 3.1. A class of dynamical systems.

**Definition 6.** [(Measured) dynamical system.] *A dynamical system  $\mathcal{S} = (\mathcal{I}, T)$  is defined by four elements*

- (a) *an alphabet  $\Sigma$  (at most) denumerable*
- (b) *a topological partition  $(\mathcal{I}_w)_{w \in \Sigma}$  of the unit interval  $\mathcal{I} = [0, 1]$ , namely a family of disjoint open intervals  $(\mathcal{I}_w)_{w \in \Sigma}$  whose closures  $\overline{\mathcal{I}_w}$  cover the interval  $\mathcal{I}$ .*
- (c) *a coding map  $\sigma$  which is constant and equal to  $w$  on each  $\mathcal{I}_w$ ,*
- (d) *a map  $T$  defined on the union of open intervals  $(\mathcal{I}_w)_{w \in \Sigma}$ , for which each restriction  $T|_{\mathcal{I}_w} : \mathcal{I}_w \rightarrow ]0, 1[$  is strictly monotonic, surjective, of class  $\mathcal{C}^2$ . The inverse branch  $h_w : ]0, 1[ \rightarrow \mathcal{I}_w$  of the branch  $T|_{\mathcal{I}_w}$  is extended into a map  $\mathcal{I} \rightarrow \overline{\mathcal{I}_w}$  also denoted by  $h_w$ , that is thus of Class  $\mathcal{C}^2$ .*

(e) *When  $\mathcal{I}$  is endowed with a probability measure  $\mu$ , with a strictly positive density  $\phi \in \mathcal{C}^1([0, 1])$ , this gives rise to a measured dynamical system  $(\mathcal{I}, T, \mu)$ .*

*The dynamical system  $(\mathcal{I}, T)$  is called the underlying dynamical system of the measured dynamical system  $(\mathcal{I}, T, \mu)$*

There are thus three main hypotheses in the definition: the surjectivity hypothesis asked for the branch  $T|_{\mathcal{I}_w}$ , and two regularity hypotheses, namely the  $\mathcal{C}^2$  regularity asked for the branches  $T|_{\mathcal{I}_w}$  and the  $\mathcal{C}^1$  regularity asked for the density  $\phi$ . Whereas it is often asked in this context that measure  $\mu$  be invariant by  $T$ , i.e.,

$$\mu(T^{-1}G) = \mu(G) \text{ for any measurable set } G,$$

this is not the case here, and, the only hypothesis asked for measure  $\mu$  is the hypothesis 6(e) on its density  $\phi$ .

The coding map  $\sigma$  is only defined on the set  $\mathcal{I} \setminus C$  where  $C = \{c_w \mid w \in \Sigma\}$  gathers the subdivision points of the partition  $(\mathcal{I}_w)$ . Each subset  $T^{-n}(C)$  is at most denumerable, and it is the same for their union that is of measure 0 for the probability  $\mu$  with a density  $\phi \in \mathcal{C}^1$ . Then, the subsets

$$\mathcal{I}_k^* = \mathcal{I} \setminus \bigcup_{n=0}^k T^{-n}(C), \quad \mathcal{I}^* = \mathcal{I} \setminus \bigcup_{n \geq 0} T^{-n}(C),$$

are measurable sets with a  $\mu$ -measure equal to 1. Furthermore, the trajectory  $\mathcal{T}(x) = (x, Tx, \dots, T^k(x), \dots)$  of  $x \in \mathcal{I}^*$  under the dynamical system  $(\mathcal{I}, T)$ , gives rises to the encoded trajectories (truncated or not)

$$M_k(x) = (\sigma(x), \sigma(Tx), \dots, \sigma(T^k(x))), \quad M(x) = (\sigma(x), \sigma(Tx), \dots, \sigma(T^n(x)), \dots).$$

The map  $M_k$  is defined on  $\mathcal{I}_k^*$ , and, due to the surjectivity hypothesis (d), the image  $M_k(\mathcal{I}_k^*)$  is the whole set  $\Sigma^k$ , and  $M : \mathcal{I}^* \rightarrow \Sigma^{\mathbb{N}}$  is a surjection. Moreover, there is a semi-conjugacy

$$(19) \quad T \circ M(x) = M \circ T(x), \quad (x \in \mathcal{I}^*)$$

between the shift  $T$  on infinite words defined in Section 2.1 and the map  $T$  that defines the dynamical system : this is why they are denoted (with a slight abuse of notation) by the same letter  $T$ .

With a prefix  $w = w_1 w_2 \dots w_k \in \Sigma^k$ , we associate the map  $h_w = h_{w_1} \circ h_{w_2} \circ \dots \circ h_{w_k}$ ; with Item(d) of Definition 6, the map  $h_w$  is well defined on  $\mathcal{I}$  and the (closed) interval  $\mathcal{I}_w = h_w(\mathcal{I})$  is called the fundamental interval relative to prefix  $w$ .

Denote by  $\mathcal{I}_w^*$  the set which gathers the reals  $x$  for which the encoded trajectory  $M(x)$  begins with the word  $w$ . Then, the set  $\mathcal{I}_w^*$  is a borelian set which differs from  $\mathcal{I}_w$  by a denumerable set of points, and, as the interval  $\mathcal{I}$  is endowed with a probability  $\mu$  that possesses a density  $\phi > 0$  of Class  $\mathcal{C}^1$ , then the probability  $p(w) = p_\mu(w)$  that the encoded trajectory  $M(x)$  begins with  $w$  satisfies  $p(w) = \mu(\mathcal{I}_w^*) = \mu(\mathcal{I}_w)$ .

Finally, we deal with the sequence  $\mathcal{P}$  of fundamental partitions,

$$(20) \quad \mathcal{P} = (\mathcal{P}_k), \quad \mathcal{P}_k = \{\mathcal{I}_w \mid w \in \Sigma^k\}, \quad \mathcal{I}_w = h_w(\mathcal{I}).$$

We now compare the two settings, the present setting and the setting of Section 2.1. For a clearer comparison, we compare the notions of Section 2.1 defined in (5), that are now underlined, with their present analogs in (20).

**Proposition 7.** *When a measured dynamical system  $(\mathcal{I}, T, \mu)$  fulfills hypotheses of Definition 6, it creates a source  $(\Sigma^{\mathbb{N}}, \underline{\mu})$ , as in Section 2, denoted as  $M(\mathcal{I}, T, \mu)$ . The measure  $\underline{\mu}$  is completely defined by the system  $(\mathcal{I}, T, \mu)$ .*

*Proof.* The set  $\mathcal{I}^*$  is endowed with  $\mu$ , and we use the surjective map  $M : \mathcal{I}^* \rightarrow \Sigma^{\mathbb{N}}$  and the semi-conjugacy in (19) to define a measure  $\underline{\mu}$  on  $\Sigma^{\mathbb{N}}$  for which the map

$$M : (\mathcal{I}^*, \mu) \rightarrow (\Sigma^{\mathbb{N}}, \underline{\mu})$$

relates the measures of fundamental intervals via the equalities,  $\underline{\mu}(\mathcal{I}_w) = \mu(\mathcal{I}_w)$ . Then, the measure  $\underline{\mu}$  is now defined on the fundamental domains  $\underline{\mathcal{I}}_w$  and extended on  $\Sigma^{\mathbb{N}}$  with Kolmogorov's Theorem.  $\square$

In the sequel, due to the semi-conjugacy (19), we consider and study Shannon weights and entropy rate directly on measured dynamical systems, rather on the dynamical sources they define. We now give main examples:

(a) The Shannon weight (of depth  $n$ ) of the measured dynamical system  $(\mathcal{I}, T, \mu)$ , denoted as  $\underline{m}_\mu(n)$ , is (by definition) the Shannon weight (of depth  $n$ ) of the dynamical source  $M(\mathcal{I}, T, \mu) = (\Sigma^{\mathbb{N}}, \underline{\mu})$ .

(b) The entropy rate of the measured dynamical system  $(\mathcal{I}, T, \mu)$ , denoted as  $\mathcal{E}_\mu$  is (by definition) – and when it exists – the entropy rate  $\mathcal{E}_{\underline{\mu}}$  of the dynamical source  $M(\mathcal{I}, T, \mu) = (\Sigma^{\mathbb{N}}, \underline{\mu})$ .

(c) The Shannon weight and the entropy rate of the measured dynamical system  $(\mathcal{I}, T, \mu)$  write in terms of the measures  $\mu(\mathcal{I}_w) = \underline{\mu}(\underline{\mathcal{I}}_w)$ ,

$$(21) \quad \underline{m}_\mu(n) = \sum_{w \in \Sigma^n} \mu(\mathcal{I}_w) |\log \mu(\mathcal{I}_w)|, \quad \mathcal{E}_\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \underline{m}_\mu(n).$$

**3.2. Generating dynamical systems.** The dynamical source  $M(\mathcal{I}, T, \mu)$  is “created” from the measured dynamical system  $(\mathcal{I}, T, \mu)$ . With the mapping  $M$ , the properties of the dynamical system relative to the measure  $\mu$  are given to the dynamical source. But, in this way, it is not clear how to compare – in a topological way – the dynamical source  $M(\mathcal{I}, T, \mu)$  with the measured dynamical system  $(\mathcal{I}, T, \mu)$  itself. We indeed need a supplementary condition:

**Definition 8.** [Generating dynamical systems] *A dynamical system  $\mathcal{S} = (\mathcal{I}, T)$  is generating if the sequence  $\mathcal{P}$  of its fundamental partitions defined in (20) generates the Borelian  $\sigma$ -algebra  $\mathcal{F}_{\mathcal{I}}$ .*

Note that the analog property always holds in the context of Section 2: this is due to the definition of the  $\sigma$ -algebra  $\mathcal{F}$  and the definition of the sequence  $\mathcal{P}$  in (5).

**Lemma 9.** *For any  $x \in \mathcal{I}^*$ , and any  $n \geq 1$ , consider the prefix  $x_{\langle n \rangle}$  of length  $n$  of  $M(x)$ , the fundamental interval  $\mathcal{I}_n(x)$  relative to the prefix  $x_{\langle n \rangle}$ , and denote by  $\delta_n(x)$  the diameter of  $\mathcal{I}_n(x)$ .*

*The three assertions are equivalent:*

- (i) *The dynamical system  $(\mathcal{I}, T, \mu)$  is generating ;*
- (ii) *The sequence  $\eta_n = \sup\{\delta(\mathcal{I}_w) \mid w \in \Sigma^n\}$  tends to 0 as  $n \rightarrow \infty$  ;*
- (iii) *For any  $x \in \mathcal{I}^*$ , the sequence  $\delta_n(x)$  tends to 0 as  $n \rightarrow \infty$ .*

Remark that Assertion (iii) seems weaker than Assertion (ii). But the two assertions are indeed equivalent due to Dini’s Theorem.

**Proposition 10.** *If the dynamical system  $\mathcal{S} = (\mathcal{I}, T)$  is generating, then:*

- (a) *There exists a continuous map  $K : \Sigma^{\mathbb{N}} \rightarrow \mathcal{I}^*$  for which  $K \circ M(x) = x$ .*
- (b) *There is an homeomorphism between the two topological spaces –  $\mathcal{I}^*$  endowed with the usual Borel topology, –  $\Sigma^{\mathbb{N}}$  endowed with the product topology of Section 2.*

*Sketch of proof.* (see [PY98] page 42). With  $\alpha \in \Sigma^{\mathbb{N}}$ , written as  $\alpha = M(x)$ , associate the subset  $K(\alpha) \subset \mathcal{I}$

$$K(\alpha) = \bigcap_{n \geq 1} \mathcal{I}_n(x).$$

Now, the following equivalence holds for any pair  $(\alpha, x)$  with  $\alpha = M(x)$ ,

$$K(\alpha) = \{x\} \iff \text{the sequence } \delta_n(x) \rightarrow 0 \text{ as } n \rightarrow \infty .$$

When the system is generating, this entails that  $K$  is well defined on  $\Sigma^{\mathbb{N}}$ , and satisfies  $K \circ M(x) = x$ . Then  $K$  is the inverse mapping of  $M$ .  $\square$

In the sequel, we will deal a priori with any dynamical source associated (as in Proposition 7) with a dynamical system –generating or not–. However, generating dynamical systems  $\mathcal{S} = (\mathcal{I}, T)$  will be highly interesting here in the context of entropy. Indeed, the homeomorphism of Proposition 10 allows us to use the dynamical point of view in the computation of entropy. This is described in the following proposition that plays an important role in the sequel:

**Proposition 11.** *When a generating dynamical system  $\mathcal{S} = (\mathcal{I}, T)$  admits an invariant probability measure  $\mu$ , then the entropy (rate) of the dynamical source  $M(\mathcal{I}, T, \mu)$  coincides with the metric entropy of the dynamical system  $(\mathcal{I}, T)$ , with respect to its invariant measure  $\mu$ , denoted as  $\mathcal{E}_\mu(\mathcal{S})$ .*

*Proof.* See [PY98] pages 80-90. □

**3.3. Dynamical systems in the Class  $\mathcal{DR}$ .** Consider now the particular case of Definition 6 when the alphabet is  $\Sigma = \{0, 1\}$ .

**Definition 12.** *Consider a binary dynamical system  $(\mathcal{I}, T)$  fulfilling the hypotheses of Definition 6, the (topological) partition of  $\mathcal{I}$  into two subintervals  $\mathcal{I}_0 = ]0, c[$  and  $\mathcal{I}_1 = ]c, 1[$  (with  $c \in ]0, 1[$ ), and the inverse branches  $a, b$  (of Class  $\mathcal{C}^2$ ) of the direct surjective branches  $A, B$  of  $T$ .*

*The system is tent shaped if the branch  $a$  is increasing and the branch  $b$  decreasing, with  $a(0) = 0, a(1) = c, b(0) = 1, b(1) = c$ .*

*The system  $(\mathcal{I}, T)$  belongs to the class  $\mathcal{DR}$  (for dynamical and recurrent) if the branches  $a$  and  $b$  moreover satisfy*

$$\forall x \in [0, 1], \quad a'(x) > 0, \quad b'(x) < 0; \quad \forall x \in ]0, 1[, \quad a'(x) < 1, \quad -1 < b'(x).$$

*When the branch  $a$  satisfies the condition  $a'(0) = 1$ , the point 0 is an indifferent point for the branch  $a$  (i.e.,  $a(0) = 0, a'(0) = 1$ ) and this leads to the class  $\mathcal{DR}\mathcal{I}$ .*

Figure 1 represents an instance of a tent-shaped system. One finds many instances of graphs of tent-shaped systems in Section 9.5.

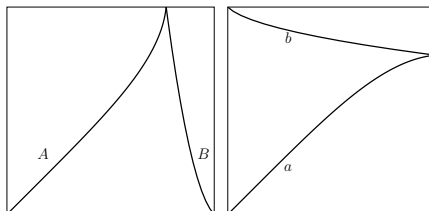


FIGURE 1. A tent-shaped dynamical system with its two direct branches (left) and its two inverse branches (right).

*Even if the system  $(I, T)$  is defined via its direct branches  $(A, B)$ , the paper mainly deals with inverse branches  $(a, b)$  and hypotheses will be asked directly on the inverse branches, as in the previous Definition 12.*

The Mean Value formulae, valid for any  $(x, y)$  with  $0 \leq x < y \leq 1$ , involve  $z$  and  $t \in ]x, y[$ , and

$$|a(y) - a(x)| = a'(z)(y - x), \quad |b(y) - b(x)| = |b'(t)|(y - x).$$

Then, the reals  $t, z$  are in  $]0, 1[$ , and  $0 < a'(z) < 1, 0 < |b'(t)| < 1$ . This entails that the inverse branches  $a, b$  of depth one satisfy strict Lipschitz conditions,

$$(22) \quad \forall (x, y), \quad 0 \leq x < y \leq 1 \quad 0 < |a(y) - a(x)| < y - x, \quad 0 < |b(y) - b(x)| < y - x.$$

With an easy recursion, any inverse branch  $h_w$  (of any depth) is weakly contracting and satisfies the inequality

$$|h_w(x) - h_w(y)| \leq |x - y|, \quad 0 \leq x \leq y \leq 1.$$

For a system  $(\mathcal{I}, T)$  of the Class  $\mathcal{DR}$ , where the absolute values  $|a'(0)|, |b'(0)|$  are strictly less than 1, the real

$$\rho = \max\{\sup_{[0,1]} |a'|, \sup_{[0,1]} |b'|\} \text{ is strictly less than 1.}$$

In this case, for any measure  $\mu$  that fulfills Def 6 (e), the measure  $\mu(\mathcal{I}_w)$  satisfies  $\mu(\mathcal{I}_w) \leq C\rho^n$ , for some constant  $C > 0$  and the entropy rate  $\mathcal{E}_\mu$  of the measured dynamical system  $(\mathcal{I}, T, \mu)$  satisfies  $\mathcal{E}_\mu \geq |\log \rho|$ . Such a system is not really interesting in our context which focuses on systems of zero entropy.

This is why we will focus in the sequel on systems of the class  $\mathcal{DR}\mathcal{I}$  for which the branch  $a$  admits an indifferent point at  $x = 0$ , (i.e., satisfies the conditions  $a(0) = 0, a'(0) = 1$ ). In this case, the derivatives  $(a^n)'(0)$  equal 1 for any  $n$ . There are two subcases, the case  $\mathcal{DR}\mathcal{I}_s$  (where  $a'(0) = 1, b'(0) > -1$ ) or the case  $\mathcal{DR}\mathcal{I}_w$  (where  $a'(0) = 1, b'(0) = -1$ ). In particular, the case  $\mathcal{DR}\mathcal{I}_w$  occurs for an emblematic instance, the Farey source (See Section 3.8).

**3.4. Recurrence.** The Class  $\mathcal{DR}$  plays a central role in the recurrence setting, as it has been described in Section 2.4.

**Proposition 13.** *Consider a dynamical system  $(I, T)$  of the Class  $\mathcal{DR}$  and the sequence  $q(n) = a^n(1)$ . Denote by  $\tau$  the Lebesgue measure. Then*

- (i) *The set  $[W > n]$  coincides with the interval  $[0, q(n)[$  and the equality  $q_\tau(n) = q(n) = a^n(1)$  holds between the wtd  $q_\tau(n)$  with respect to the Lebesgue measure and  $q(n)$ . The level set  $[W = m]$  is an interval of  $\mathcal{I}$  whose interior is  $\mathcal{J}_m = ]q(m), q(m-1)[$ .*
- (ii) *The sequence  $n \mapsto q(n)$  tends to 0 (when  $n \rightarrow \infty$ )*
- (iii) *Any measured dynamical system  $(\mathcal{I}, T, \mu)$  associated with the dynamical system  $(I, T)$  defines a recurrent dynamical source  $M(\mathcal{I}, T, \mu)$ .*

*Proof.* (i) The set  $[W > n] = \mathcal{I}_{0^n}$  is written in terms of iterates of the inverse branch  $a$  as  $[W > n] = ]0, a^n(1)[$ .

(ii) For any  $x \in ]0, 1]$ , the sequence  $a^n(x)$  tends to 0: by induction, indeed, the sequence  $x \mapsto a^n(x)$  is non zero and strictly decreasing, due to (22). It is thus convergent and, as  $a$  is continuous, its limit is the only fixed point for  $a$ , namely 0.

(iii) Any interval  $\mathcal{I}_{w \cdot 0^n}$  associated with a word  $w \in \{0, 1\}^*$  is written as

$$\mathcal{I}_{w \cdot 0^n} = h_w(\mathcal{I}_{0^n}) = h_w([W > n]) = [h_w(0), h_w(q(n))].$$

As  $\phi \in \mathcal{C}^1$ , the function  $F$  given by  $F(x) = \int_0^x \phi(t)dt$  belongs to  $\mathcal{C}^2([0, 1])$ , and

$$\begin{aligned} \mu(\mathcal{I}_{w \cdot 0^n}) &= \mu(h_w(\mathcal{I}_{0^n})) = \int_{h_w(0)}^{h_w(q(n))} \phi(u)du \\ &= F(h_w(q(n))) - F(h_w(0)) = (h_w(q(n)) - h_w(0)) F'(x_w) \end{aligned}$$

with  $x_w \in ]h_w(0), h_w(q(n))]$ .

As  $F$  is of Class  $\mathcal{C}^2$ , one has  $|F'(x)| \leq A$  for any  $x \in \mathcal{I}$  and some  $A$ . Moreover, the inverse branch  $h_w$  is weakly contracting and  $|h_w(0) - h_w(q(n))| \leq q(n)$ . Then the inequality

$$\mu(\mathcal{I}_{w \cdot 0^n}) \leq Aq(n)$$

holds. Thanks to (ii), as  $n$  tends to  $\infty$ , the measure  $\mu(\mathcal{I}_{w \cdot 0^n})$  tends to 0. With Definition 3, the source  $M(\mathcal{I}, T, \mu)$  is recurrent.  $\square$

It is now possible to “transfer” the definitions of Section 2.4 into the context of Class  $\mathcal{DR}$ .

- The wtd  $q_\mu(n) = \mu[W > n]$  of the measured dynamical system  $(\mathcal{I}, T, \mu)$  is (by definition) the wtd  $\underline{\mu}[W > n]$  of the dynamical source  $M(\mathcal{I}, T, \mu)$ , defined in (17).
- This defines the generating function  $Q_\mu(v)$  of the measured dynamical system  $(\mathcal{I}, T, \mu)$  as the generating function  $Q_{\underline{\mu}}(v)$  of the dynamical source  $M(\mathcal{I}, T, \mu)$  defined in (18).

**3.5. Block source.** We now associate with a dynamical system  $(\mathcal{I}, T)$  of the Class  $\mathcal{DR}$  another dynamical system, denoted as  $(\mathcal{I}, \widehat{T})$ , called the block dynamical system of  $(\mathcal{I}, T)$ , and defined in the context of Definition 6 as follows. It uses the level sets defined in Proposition 13 and the fact that the sequence  $q(n)$  tends to 0 ( $n \rightarrow \infty$ ) (see Proposition 13(ii)).

**Definition 14.** [Block dynamical system] *When the system  $(\mathcal{I}, T)$  belongs to the Class  $\mathcal{DR}$ , the block dynamical system  $(\mathcal{I}, \widehat{T})$  is described in the context of Definition 6 as follows:*

- (a) *the alphabet is  $\mathbb{N}$*
- (b) *the topological partition is defined by the intervals  $\mathcal{J}_m = ]q(m), q(m+1)[$  for  $m \geq 1$ , and the set of subdivision points is  $\widehat{C} = \{q(n) \mid n \geq 0\} \cup \{0\}$ .*
- (c) *the coding map  $\widehat{\sigma}$  is constant on each  $\mathcal{J}_m$  and equals  $m$ .*
- (d) *the map  $\widehat{T}$  is defined on the union of intervals  $\mathcal{J}_m$  for  $m \geq 1$ . The inverse of the restriction  $\widehat{T}_m : \mathcal{J}_m \rightarrow ]0, 1[$  is  $g_m : ]0, 1[ \rightarrow \mathcal{J}_m$ , that is extended into a map  $\mathcal{I} \rightarrow [q(m), q(m+1)]$  equal to  $g_m = a^{m-1} \circ b$ .*

The following proposition (whose proof is clear and thus omitted) now relates the two notions: block dynamical systems, just defined here and block sources of Section 2, defined in Definition 5.

**Proposition 15.** [Block source in the context of Class  $\mathcal{DR}$ ] *Consider a measured dynamical system  $(\mathcal{I}, T, \mu)$  whose underlying system  $(\mathcal{I}, T)$  belongs to the Class  $\mathcal{DR}$ . Then, the dynamical source  $M(\mathcal{I}, \widehat{T}, \mu)$  is the block source of the source  $M(\mathcal{I}, T, \mu)$ .*

Just like in Section 3.1, the block dynamical system admits a sequence  $\mathcal{Q} = (\mathcal{Q}_n)$  of fundamental partitions, with

$$(23) \quad \mathcal{Q}_n = \{g_{m_1} \circ g_{m_2} \circ \dots \circ g_{m_n}(\mathcal{I}) \mid m_i \geq 1\}.$$

The following result describes the impact of the behaviour of the block system on the system itself.

**Proposition 16.** *Consider a dynamical system  $\mathcal{S} = (\mathcal{I}, T)$  of the class  $\mathcal{DR}$ , and its associated block dynamical system  $\mathcal{B} = (\mathcal{I}, \widehat{T})$ . If the block dynamical system  $\mathcal{B}$  is generating, then the dynamical system  $\mathcal{S}$  is itself generating.*

*Proof.* We consider an element of  $\mathcal{I}^*$  and the sequence  $\delta_n(x)$  defined in Lemma 9. Using Lemma 9(iii), we have to prove that the sequence  $\delta_n(x)$  tends to 0. As the sequence  $n \mapsto \delta_n(x)$  is weakly decreasing, it is enough to build a subsequence of the sequence  $\delta_n(x)$  that tends to 0. Such a subsequence is associated with a subsequence  $w_{(n)}$  of prefixes of  $M(x)$  for which  $\delta(\mathcal{I}_{w_{(n)}})$  tends to 0.

As the infinite word  $M(x)$  contains an infinite number of ones, located at indices  $i_j$  with a strictly increasing sequence  $j \mapsto i_j$ , the sequence  $i_n$  tends to  $\infty$  as  $n \rightarrow \infty$ . Consider, for any  $n$ , the smallest prefix  $w_{(n)}$  of  $M(x)$  that contains a number of ones equal to  $n$ . The prefix  $w_{(n)}$  is thus written as a concatenation of  $n$  blocks, and, as the block system  $\mathcal{B}$  is generating, with Lemma 9 Item (ii),  $\lim_{n \rightarrow \infty} \delta(\mathcal{I}_{w_{(n)}}) = 0$ . This ends the proof  $\square$

**3.6. The Good Class.** For a dynamical system  $(\mathcal{I}, T)$  of the class  $\mathcal{DR}$ , each inverse branch  $g_m = a^{m-1} \circ b : \mathcal{I} \rightarrow \mathcal{J}_m$  of the block dynamical system is surjective of class  $\mathcal{C}^2$ . But we need stronger properties for the block dynamical system, and we ask  $(\mathcal{I}, \widehat{T})$  to belong to the Good Class, which has been already strongly used in [BV05] and in [CV15]. The definition is now recalled:

**Definition 17** (Good Class). *Consider a (tent-shaped) dynamical system  $\mathcal{S}$ , its block system  $\mathcal{B}$ , its block fundamental intervals  $\mathcal{J}_m$ , the set  $\mathcal{G} := \{g_m = a^{m-1} \circ b \mid m \geq 1\}$  of inverse branches of depth one, and the set  $\mathcal{G}^n$  of inverse branches of depth  $n$ .*

*The block dynamical system  $\mathcal{B}$  belongs to the Good Class if the following holds*

- (a) *The convergence abscissa of its Dirichlet series  $\Delta(s) = \sum_{m \geq 1} |\mathcal{J}_m|^s$  is strictly less than 1.*
- (b) *The system  $\mathcal{B}$  is strictly contracting : there exists a contraction ratio  $\eta < 1$   $\eta = \limsup_{n \rightarrow \infty} \eta_n$ , with  $\eta_n = \max\{|g'(x)| \mid g \in \mathcal{G}^n, x \in I\}$ .*
- (c) *Its inverse branches  $g_m = a^{m-1} \circ b$  (of depth one) satisfy the bounded distortion property,*

$$(24) \quad \exists L \geq 1, \forall m \geq 1, \forall (x, y) \in \mathcal{I}^2, \quad \text{one has} \quad \frac{1}{L} \leq \left| \frac{g'_m(x)}{g'_m(y)} \right| \leq L.$$

There are some important remarks.

– First, a block system of the Good Class is generating: this is due to Property 17(b) and Lemma 9(ii).

– Second, we will prove later on, in Section 8.4, that the block system of a system of Class  $\mathcal{DR}$  already satisfies Item(b) – moreover, in a strong sense–. Then, when we ask a system of the Class  $\mathcal{DR}$  to have a block system of the Good Class, the remaining conditions to fulfill are Conditions (a) and (c) of Definition 17. They play an important role in dynamical analysis of Sections 5 and 6.

**3.7. The Class  $\mathcal{DRBGC}$ .** We now define our main class of interest:

**Definition 18.** [Class  $\mathcal{DRBGC}$ ] *This is the class of measured dynamical systems  $(I, T, \mu)$  of Definition 6 for which*

- (a) *the system  $(I, T)$  belongs to the class  $\mathcal{DRI}$  (see Definition 12);*
- (b) *the block system  $(I, \widehat{T})$  belongs to the Good Class (see Definition 17).*

We deal with the Class  $\mathcal{DRI}$  in Item (a), because we are mainly interested in systems with zero entropy (See Section 3.3). We recall that there is no hypothesis asked for the measure  $\mu$  except that it fulfills hypothesis 6(e) of Definition 6.

**3.8. An emblematic instance: the Farey system.** The Farey dynamical system plays an important role in Number Theory, notably because its block dynamical system coincides with the Gauss map, as we recall it later on. This also provides a natural example of dynamical system with intermittent type, that is studied for instance in Prellberg's paper [PS92]. We will see that the Farey dynamical system belongs to the  $\mathcal{DRIS}$  class, introduced and studied in Section 8, and even to the Class  $\mathcal{DRIL}$  of Section 9.

The Farey map  $T$  possesses two complete inverse branches  $a$  and  $b$

$$\begin{cases} a : [0, 1] \rightarrow [0, 1/2] & a(x) = x/(1+x), \\ b : [0, 1] \rightarrow [1/2, 1] & b(x) = 1/(1+x). \end{cases}$$

As the iterates  $a^n$  of the first inverse branch  $a$  satisfy  $a^n(x) = x/(1+nx)$ , the sequence  $q(n) = a^n(1)$  is  $q(n) = 1/(n+1)$  and  $(a^n)'(x) = 1/(1+nx)^2$ . Moreover,

as the derivative  $b'$  of the second inverse branch  $b$  satisfies  $b'(x) = -1/(1+x)^2$ , the Farey source thus belongs to the class  $\mathcal{DR}\mathcal{L}_w$ .

The block system of the Farey system is defined via the partition  $\mathcal{J}_m = [q(m), q(m-1)]$  for  $m \geq 1$  with  $q(m) = 1/(m+1)$  and the local inverses  $g_m$  are  $g_m = a^{m-1} \circ b$ . For any  $m \geq 1$ , the branch  $a^{m-1}(x)$  equals  $x/(1+(m-1)x)$ , and the equality

$$a^{m-1} \circ b(x) = \frac{b(x)}{1+(m-1)b(x)} = \left(\frac{1}{1+x}\right) \left(\frac{1+x}{1+x+(m-1)}\right) = \frac{1}{m+x}$$

proves that  $a^{m-1} \circ b$  coincides with the linear fractional transformation (LFT in shorthand)  $g_m : x \mapsto 1/(m+x)$  that is indeed the inverse of the Gauss map on the interval  $\mathcal{J}_m$ . One has thus proven that the block map  $\widehat{T}$  of the Farey map is the Gauss map,

$$\widehat{T}(0) = 0, \quad \widehat{T}(x) = \left\{ \frac{1}{x} \right\} \quad (x \neq 0).$$

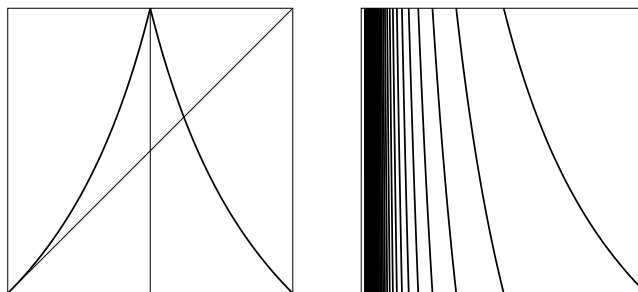


FIGURE 2. The Farey map and the Gauss map

#### 4. STATEMENT OF THE FOUR MAIN RESULTS.

This Section states the four main results of the paper that are all obtained for a source of the Class  $\mathcal{DR}\mathcal{B}\mathcal{G}\mathcal{C}$  introduced in Section 3.7. The first main result (Theorem 19) describes invariant densities, of the block source, and of the source itself. Section 4.2 states the second main result of the paper (Theorem 20) that relates the generating functions  $N_\mu$  and  $M_\mu$  with the generating function  $Q_\mu$  associated with the distribution of the waiting time. It can be viewed as a renewal result. Singularity analysis, described in Section 4.3, performs a round trip between singularities of generating functions and their coefficients. It deals in a natural way with slowly varying functions precisely described in Section 7.1. Section 4.5 returns to the mean values of interest, the mean number of ones and the Shannon weights, and obtains the third main result, stated as Theorem 23. Section 4.6 describes the path for obtaining sources with prescribed Shannon weights, and gives a summary of main results obtained later on in Sections 8 and 9.

##### 4.1. The first main result: invariant densities.

**Theorem 19.** *Consider a measured dynamical system  $(I, T, \mu)$  of the Class  $\mathcal{DR}\mathcal{B}\mathcal{G}\mathcal{C}$ , and denote by  $W$  the waiting time of symbol one. The following holds*

- The dynamical system  $(I, \widehat{T})$  has an invariant probability measure  $\nu$  with a strictly positive density  $\psi$  in  $\mathcal{C}^1([0, 1])$  and a finite entropy  $\mathcal{E}_\nu(\mathcal{B})$ .*
- The dynamical system  $(I, T)$  admits an invariant function  $\phi_0$ , whose associated measure  $\int_{\mathcal{I}} \phi_0(t) dt$  equals  $\mathbb{E}_\nu[W]$ . When the mean  $\mathbb{E}_\nu[W]$  is finite, the measure  $\pi$  associated with  $\rho = (1/\mathbb{E}_\nu[W])\phi_0$  is an invariant probability measure and  $\rho$  satisfies  $\lim_{x \rightarrow 0} \rho(x) = \infty$ .*

Theorem 19 will be proven in Section 5.

**4.2. The second main result: renewal equations.** We study two costs in parallel: the Shannon weight  $m$  defined in Section 2.2, and the number  $n$  of ones introduced in Section 2.3. The following theorem relates the behaviours of the four generating functions of interest  $N_\mu(v)$ ,  $M_\mu(v)$ ,  $Q_\mu(v)$ ,  $Q_\nu(v)$ , as the real  $v \rightarrow 1^-$ .

**Theorem 20.** [Generating functions  $N_\mu(v)$  and  $M_\mu(v)$ ] *Consider a measured dynamical system  $\mathcal{S} = (\mathcal{I}, T, \mu)$  of the class  $\mathcal{DRBGC}$ , the invariant measure  $\nu$  of the block system  $\mathcal{B} = (I, \hat{T})$ , and its finite entropy  $\mathcal{E}_\nu(\mathcal{B})$ . Denote by  $W$  the waiting time of symbol one, and by  $Q_\nu(v)$  the generating functions of the wtd sequence defined in (18) relative to the invariant measure  $\nu$  of the block system  $(\mathcal{I}, \hat{T})$ .*

*Consider the two generating functions  $N_\mu(v)$  of number of ones,  $M_\mu(v)$  of Shannon weights— resp defined, as in (8), from the number  $n$  of ones introduced in Section 2.3 and the Shannon weight  $m$  defined in Section 2.2.*

*The following holds, as the real  $v$  tends to  $1^-$ ,*

(A) *the dominant behaviours of the two generating functions  $N_\mu(v)$  and  $M_\mu(v)$  are related to the dominant behaviour of the generating function  $Q_\nu(v)$ ,*

$$(25) \quad (1-v)N_\mu(v) \sim_{v \rightarrow 1^-} \frac{1}{(1-v)Q_\nu(v)}, \quad (1-v)M_\mu(v) \sim_{v \rightarrow 1^-} \frac{\mathcal{E}_\nu(\mathcal{B})}{(1-v)Q_\nu(v)}.$$

(B) *Consider the generating function  $Q_\mu(v)$  of the wtd sequence relative to  $\mu$ ; then*

$$(26) \quad \text{the ratio } \frac{Q_\mu(v)}{Q_\nu(v)} \text{ has a finite limit } D_\mu \text{ as } v \rightarrow 1^- \text{ and}$$

$$(27) \quad (1-v)N_\mu(v) \sim_{v \rightarrow 1^-} \frac{D_\mu}{(1-v)Q_\mu(v)}, \quad (1-v)M_\mu(v) \sim_{v \rightarrow 1^-} \frac{D_\mu \mathcal{E}_\nu(\mathcal{B})}{(1-v)Q_\mu(v)}.$$

The proof of Theorem 20 will be done in Section 6. We first remark that the dominant behaviours in (25) do not depend on the measure  $\mu$ . We also give an interpretation of the first relation in (27) that can be viewed as a renewal equation between the two generating functions  $(1-v)Q_\mu(v)$  and  $(1-v)N_\mu(v)$  and the partial sums of their coefficients.

With Lemma 1, the partial sums of coefficients of the two generating functions  $(1-v)Q_\mu(v)$  and  $(1-v)N_\mu(v)$  are resp. equal to  $q_\mu(n)$  and  $\underline{n}_\mu(n)$  with

- (a)  $q_\mu(n)$  = the probability that the waiting time of the set  $\mathcal{J}$  is at least equal to  $n$ ,
- (b)  $\underline{n}_\mu(n)$  = the mean value of the time spent in the set  $\mathcal{J}$  during the first  $n$  steps.

We now express these gf's in terms of the set  $\mathcal{J}$  which gathers the infinite words which begin with the symbol one. The relations hold

$$r_\mu(k) = q_\mu(k-1) - q_\mu(k) = \mu [T^{-k} \mathcal{J} - (\cup_{\ell < k} T^{-\ell} \mathcal{J})], \quad \underline{n}_\mu(k) - \underline{n}_\mu(k-1) = \mu [T^{-k} \mathcal{J}],$$

and entail the two equalities

$$(1-v)Q_\mu(v) = 1 - \sum_{k \geq 1} r_\nu(k)v^k = 1 - \sum_{k \geq 1} \mu [T^{-k} \mathcal{J} - (\cup_{\ell < k} T^{-\ell} \mathcal{J})] v^k,$$

$$(1-v)N_\mu(v) = \underline{n}_\mu(0) + \sum_{k \geq 1} (\underline{n}_\mu(k) - \underline{n}_\mu(k-1))v^k = \sum_{k \geq 1} \mu [T^{-k} \mathcal{J}] v^k.$$

Then, the renewal equation is

$$(28) \quad \left( \sum_{k \geq 1} \mu [T^{-k} \mathcal{J}] v^k \right) \cdot \left( 1 - \sum_{k \geq 1} \mu [T^{-k} \mathcal{J} - (\cup_{\ell < k} T^{-\ell} \mathcal{J})] v^k \right) \sim_{v \rightarrow 1^-} D_\mu.$$

In particular, when the initial measure is the invariant measure of the block dynamical system, the ratio  $D_\mu$  equals 1.

Aaronson [A81] obtains (another) renewal equation through the notion of Darling-Kac sets that is described in Section 6. In the present paper, we directly obtain the previous renewal relation with the derivatives of the trivariate function  $\Lambda_\mu$ .

**4.3. Singularity analysis: Towards the third main result.** The general singularity analysis process relates the asymptotics of a sequence and the behaviour of its generating function near the dominant singularity. Generally speaking, complex analysis is very often needed. However, in the present context, due to strong positivity properties, it is enough to deal with real analysis, as we now explain.

We perform a “round trip” between the behaviour (as  $n \rightarrow \infty$ ) of a positive sequence  $q_\mu(n)$  and the behaviour (as  $v$  real tends to  $1^-$ ) of its generating function  $Q_\mu(v)$  that we now describe more precisely.

We begin with a (positive) sequence  $q_\mu(n)$  and study the partial sum  $Q_n^{(\mu)} = q_\mu(0) + q_\mu(1) + \dots + q_\mu(n-1)$ . There are three steps :

- (A) [Abelian step] An Abelian Theorem describes the behaviour of the generating function  $Q_\mu(v)$  when  $v \rightarrow 1^-$  along the positive real line from the asymptotic behaviour of the sequence  $Q_n^{(\mu)}$ ;
- (B) Using Theorem 20 provides the behaviour of the two generating functions  $(1-v)N_\mu(v)$ ,  $(1-v)M_\mu(v)$  when  $v \rightarrow 1^-$  along the positive real line.
- (C) [Tauberian step] We return to the asymptotic estimate of the partial sums of coefficients of the functions  $(1-v)N_\mu(v)$ ,  $(1-v)M_\mu(v)$  –that resp. coincide with  $\underline{n}_\mu(n)$ ,  $\underline{m}_\mu(n)$ – via a Tauberian Theorem.

The following central theorem (Theorem 22) provides the Abelian step (A) via its implication [(2)  $\implies$  (1)] and the Tauberian step (C) via its implication [(1)  $\implies$  (2)]. It deals with the set  $\mathcal{SV}$  of slowly varying functions that is now defined.

**Definition 21.** A positive function  $V$  defined on  $]0, \infty[$  is said to be slowly varying

- (a) at infinity if, for any  $t > 0$ , one has  $V(tx)/V(x) \rightarrow 1$  as  $x \rightarrow \infty$ .
- (b) at 0 if  $x \mapsto V(1/x)$  is slowly varying at infinity.

The set of slowly varying functions at infinity is denoted by  $\mathcal{SV}$  and the set of slowly varying functions at zero by  $\mathcal{SV}_0$ .

The sets  $\mathcal{SV}$  and  $\mathcal{SV}_0$  of slowly varying functions play an important role in the sequel of the paper. Main examples of such functions are provided by powers (positive or negative) of logarithms. Main properties of such functions are described in Section 7.1. See also the book [F71] Section V.III.8 p. 276.

**Theorem 22.** [P04], [HL23], [K30], revisited in [F71] section XIII.5 p. 447.

Consider a real  $\rho \geq 0$ , a sequence  $q(n) \geq 0$  and its partial sums  $Q_n = q(0) + q(1) + \dots + q(n-1)$  together with a function  $U$  slowly varying at infinity. Assume that the generating function  $Q(v) = \sum q(n)v^n$  is convergent for  $v \in [0, 1[$ .

The two assertions (1) and (2), are equivalent

$$(1) \quad Q(v) \sim_{1^-} \frac{1}{(1-v)^\rho} U\left(\frac{1}{1-v}\right) \quad (2) \quad Q_n \sim_\infty \frac{1}{\Gamma(\rho+1)} n^\rho U(n).$$

This strategy of proof “with a round trip” has already been used in different contexts. The steps (A) and (C) are similar as here, but the “middle step” (B) is a relation between two generating functions that may differ from here. For instance, with a middle step (B) of another flavour, already described at the end of previous

Section and made more precise in Section 6, Aaronson obtains distributional limit theorems in Infinite Ergodic Theory [A97].

Here, the step (B) is directly obtained via operations on generating functions.

**4.4. Sequences of interest.** When dealing with Theorem 22, there are two main cases, the case when  $\mathbb{E}_\nu[W]$  is finite or the case when it is infinite.

In the case when  $\mathbb{E}_\nu[W]$  is infinite, we are interested in wtd sequences  $q_\nu(n)$  which tend to zero and whose associated series is divergent. We then consider sequences which are written in terms of sequences of the Class  $\mathcal{SV}$  as

$$(29) \quad q_\nu(n) = n^{-\beta} V_n, \quad V_n = V(n) \quad \text{where } V \text{ satisfies}$$

- (i)  $V \in \mathcal{SV}$  when  $\beta \in ]0, 1[$ ;
- (ii)  $V \in \mathcal{SV}$  satisfying  $V_n = \Theta((\log n)^\delta)$  with  $\delta > -1$  when  $\beta = 1$ ;
- (iii)  $V \in \mathcal{SV}$  with  $V_n \rightarrow 0$  and  $\sum_{k < n} V_k \rightarrow \infty$  when  $\beta = 0$ .

When we wish more explicit results, we focus on sequences  $q_\nu(n)$  that satisfy

$$(30) \quad q_\nu(n) \sim K_\nu n^{-\beta} (\log n)^\delta,$$

where the parameter  $(\beta, \delta)$  belongs to the set

$$(31) \quad \Gamma_Q = (]0, 1[ \times \mathbb{R}) \cup (\{1\} \times ]-1, +\infty[) \cup (\{0\} \times ]-\infty, 0[).$$

**4.5. The third main result.** The third main result of the paper relates estimates of number of ones and Shannon weights with estimates of the waiting time distribution:

**Theorem 23.** [Number of ones and Shannon weights] *Consider a measured dynamical system  $\mathcal{S} = (\mathcal{I}, T, \mu)$  of the class  $\mathcal{DRBGC}$ , the invariant measure  $\nu$  of the block system  $\mathcal{B} = (I, \hat{T})$ , and the finite entropy  $\mathcal{E}_\nu(\mathcal{B})$ . Denote by  $W$  the waiting time of the symbol one.*

The following holds:

(A) *In the case when  $\mathbb{E}_\nu[W]$  is infinite and the distribution  $q_\nu(n)$  of the waiting time  $W$  satisfies (29), the average number of ones  $\underline{m}_\mu(n)$  is  $o(n)$ , and the asymptotic estimates hold for the Shannon weights  $\underline{m}_\mu(n)$ ,*

$$(32) \quad \underline{m}_\mu(n) \sim \mathcal{E}_\nu(\mathcal{B}) \underline{n}_\mu(n).$$

(B) *In the case when  $\mathbb{E}_\nu[W]$  is infinite and the sequence  $q_\nu(n)$  satisfies (30), with a triple  $(K_\nu, \beta, \delta)$  that satisfies (31), then the Shannon weights admit the following asymptotic estimates*

$$(33) \quad \underline{m}_\mu(n) \sim \left\{ \begin{array}{ll} \frac{\mathcal{E}_\nu(\mathcal{B})}{K_\nu} \frac{(\delta + 1)n^\beta}{(\log n)^{\delta+1}} & (\beta = 1) \\ \frac{\mathcal{E}_\nu(\mathcal{B})}{K_\nu} \frac{1}{\Gamma(\beta + 1)\Gamma(1 - \beta)} \frac{n^\beta}{(\log n)^\delta} & (0 \leq \beta < 1) \end{array} \right\}.$$

(C) *When  $\mathbb{E}_\nu[W]$  is finite, the average number of ones and the Shannon weights are of linear order,*

$$(34) \quad \underline{n}_\mu(n) \sim \frac{1}{\mathbb{E}_\nu[W]} n, \quad \underline{m}_\mu(n) \sim \frac{\mathcal{E}_\nu(\mathcal{B})}{\mathbb{E}_\nu[W]} n,$$

*the entropy rate  $\mathcal{E}_\mu$  of the source  $(\Sigma^\mathbb{N}, \underline{\mu})$  exists, and satisfies  $\mathcal{E}_\mu = \mathcal{E}_\nu(\mathcal{B})/\mathbb{E}_\nu[W]$ .*

We first remark that all the previously described estimates do not depend on the measure  $\mu$ . Moreover, with Assertions (A) and (C), and because we are mainly interested in sources  $\mathcal{S}$  of zero entropy, we are (mainly) led to sources whose wtd sequence  $q_\nu(n)$  is associated with a series  $Q_\nu(v)$  which tends to  $\infty$  as  $v \rightarrow 1^-$ .

Second, there exists a heuristics that explains the estimate proven in Assertion (A) as follows. Consider a random prefix of length  $n$  emitted by the source  $S$  endowed with probability  $\mu$ . This prefix contains an ‘‘average’’ number of ones (and thus an ‘‘average’’ number of full blocks) equal to  $\underline{n}_\mu(n)$ . Each full block brings an ‘‘average’’ amount of information equal to  $\mathcal{E}_\nu(\mathcal{B})$ . Then the total ‘‘average’’ amount of information brought by a random prefix of length  $n$ , equal (by definition) to  $\underline{m}_\mu(n)$  satisfies (32). However, we do not succeed in transforming this heuristics into a direct probabilistic proof. In particular, as the prefix may be not a full block, its tail is not easy to deal with. The (indirect) analytic proof developed in Sections 6 and 7 is based on the dominant expressions of the generating functions  $N_\mu$  and  $M_\mu$  near  $v = 1^-$  that both involve a double quasi-inverse (see Eqns (65) and (66)).

Third, Assertion (C) cannot be applied to the invariant measure  $\pi$  of the source  $\mathcal{S} = (I, T)$  studied in Theorem 19. As it is proven in Theorem 19, its density  $\rho$  is infinite at  $x = 0$ , and does not belong to  $\mathcal{C}^1([0, 1])$ . If Assertion (C) could be applied to measure  $\pi$ , this would entail the estimate

$$m_\pi(n) \sim n \cdot \mathcal{E}_\pi(\mathcal{S}), \quad \text{and thus} \quad \mathcal{E}_\pi(\mathcal{S}) = \frac{\mathcal{E}_\nu(\mathcal{B})}{\mathbb{E}_\nu[W]},$$

the last equality being known as Abramov formula.

The proof of Theorem 23 is provided in Section 7.

**4.6. Towards the fourth result.** Consider a source  $\mathcal{S}$  of the Class  $\mathcal{DRBGC}$ , with the pair  $(\beta_Q, \delta_Q)$  of its wtd sequence that belongs to the set  $\Gamma_Q$  defined in (31). Item (B) of Theorem 23 proves that its Shannon weights are of the general form

$$(35) \quad \underline{m}_\mu(n) = \Theta(n^{\beta_M} (\log n)^{\delta_M}),$$

where the pair  $(\beta_M, \delta_M)$  is defined from the pair  $(\beta_Q, \delta_Q)$  via the injective map defined on  $\Gamma_Q$ ,

$$(36) \quad (Q \rightarrow M) : (\beta_Q, \delta_Q) \mapsto (\beta_M, \delta_M) \quad \text{with}$$

$$\beta_M = \beta_Q \quad \text{and} \quad [\delta_M = -\delta_Q \quad (\beta_Q < 1) \quad \text{or} \quad \delta_M = -(\delta_Q + 1) \quad (\beta_Q = 1)],$$

whose image is

$$(37) \quad \Gamma_M = (]0, 1[ \times \mathbb{R}) \cup (\{1\} \times ]-\infty, 0]) \cup (\{0\} \times ]0, +\infty[).$$

As we wish to exhibit a source  $\mathcal{S}$  of the class  $\mathcal{DRBGC}$  for which Shannon weights satisfy Eqn (35), our first task is to prove our three main Theorems (Theorem 19, Theorem 20 and Theorem 23). Then, if we know a source  $\mathcal{S}$  of the class  $\mathcal{DRBGC}$  with a prescribed wtd of pair  $(\beta_Q, \delta_Q)$ , we may use the mapping described in (36).

In Sections 8 and 9, we will introduce the two following classes  $\mathcal{DRIS}$  and  $\mathcal{DRIL}$  and obtain the following results, along the following main steps:

[Exhibiting a source  $\mathcal{S}$  with prescribed Shannon weights]

- (i) The  $\mathcal{DRIS}$  Class introduced in Section 8 satisfies the following:
  - (a) The inclusion  $\mathcal{DRIS} \subset \mathcal{DRBGC}$  holds [proven in Theorem 42]; .
  - (b) For a source of the  $\mathcal{DRIS}(\gamma)$  Class, with  $\gamma > 1$ , the expectation  $\mathbb{E}_\nu[W]$  is infinite. For  $\gamma < 1$ , the expectation  $\mathbb{E}_\nu[W]$  is finite [see Theorem 43];

- (ii) The class  $\mathcal{DRIL}$  introduced in Section 9 is a subclass of the class  $\mathcal{DRIS}$ . A source of the  $\mathcal{DRIL}(\gamma, \delta)$  Class has a prescribed wtd described in Proposition 48.
- (iii) Theorem 49 in Section 9 gives a final answer to our “exhibition”.

## 5. GENERATING OPERATORS AND GENERATING FUNCTIONS. THE FIRST MAIN THEOREM.

This Section aims at proving Theorem 19 with tools of dynamical analysis. The first three subsections present the main principles of dynamical analysis, where *secant* transfer operators generate probabilities. Then, Section 5.4 provides in Theorem 26 an alternative expression of the trivariate generating function  $\Lambda_\mu(v, t, s)$  for a dynamical source of Class  $\mathcal{DR}$  in terms of these secant transfer operators, together with the secant function  $S[\phi]$  associated with the density  $\phi$  of the measure  $\mu$ . In the following of the Section (Section 5.5), the main results on unweighted transfer operators are first presented, as they are obtained in previous works of Cesaratto and Vallée [CV15]. These previous works entail that a block source of the Good Class admits an invariant measure  $\nu$ ; we then focus here on the properties of the entropy, and prove that the entropy  $\mathcal{E}_\nu(\mathcal{B})$  follows the Rohlin formula. The remainder of the Section is devoted to the proof of Theorem 19(b).

**5.1. Generating operators.** For a complex number  $s \in \mathbb{C}$ , we associate with a map  $g$  of class  $\mathcal{C}^1(\mathcal{I})$  the tangent transfer operator  $\mathbf{M}_s\langle g \rangle$  defined as

$$\mathbf{M}_s\langle g \rangle[f](x) := |g'(x)|^s f(g(x)),$$

that transforms a function  $f$  defined on  $\mathcal{I}$  into the function  $\mathbf{M}_s\langle g \rangle[f]$  defined on  $\mathcal{I}$ . In the same vein, the secant transfer operator  $\mathbb{M}_s\langle g \rangle$

$$\mathbb{M}_s\langle g \rangle[F](x, y) := \left| \frac{g(x) - g(y)}{x - y} \right|^s F(g(x), g(y)),$$

transforms a function  $F$  defined on  $\mathcal{I}^2$  into another function  $\mathbb{M}_s\langle g \rangle[F]$  defined on  $\mathcal{I}^2$ , and the equality holds between the two versions of transfer operators,

$$(38) \quad \mathbb{M}_s\langle g \rangle[F](x, x) = \mathbf{M}_s\langle g \rangle[\text{Diag } F](x), \quad \text{with} \quad \text{Diag } F(x) = F(x, x).$$

We focus on a dynamical system  $\mathcal{S} = (\mathcal{I}, T)$  as defined in Definition 6, consider the case when  $g = h_w$  is an inverse branch (of any depth) associated with a word  $w$ , and use the following notations that insist on the dependence with respect to words,

$$(39) \quad \mathbb{H}_s\langle w \rangle := \mathbb{M}_s\langle h_w \rangle, \quad \mathbf{H}_s\langle w \rangle := \mathbf{M}_s\langle h_w \rangle.$$

There are now fundamental morphisms, due to relations

$$(40) \quad \mathbf{H}_s\langle u \cdot w \rangle = \mathbf{H}_s\langle w \rangle \circ \mathbf{H}_s\langle u \rangle, \quad \mathbb{H}_s\langle u \cdot w \rangle = \mathbb{H}_s\langle w \rangle \circ \mathbb{H}_s\langle u \rangle.$$

The transfer operators of the system  $(\mathcal{I}, T)$  are (with their two versions, tangent or secant)

$$(41) \quad \sum_{w \in \Sigma} \mathbf{H}_s\langle w \rangle, \quad \sum_{w \in \Sigma} \mathbb{H}_s\langle w \rangle.$$

**5.2. Quasi-inverse  $(I - \mathbb{L}_{s,t})^{-1}$ .** We now focus on binary dynamical systems of Definition 6. We often replace the symbol 0 by the symbol  $a$ , and 1 by  $b$ , in order to deal with more readable formulae, and we use the shorthand notations,

$$\mathbb{A}_s = \mathbb{H}_s\langle a \rangle, \quad \mathbb{B}_s = \mathbb{H}_s\langle b \rangle,$$

with an easy adaptation to the tangent versions. The secant transfer operators of the system are defined as the sums

$$(42) \quad \mathbb{L}_s = \mathbb{A}_s + \mathbb{B}_s, \quad \mathbb{L}_{s,t} = \mathbb{A}_s + t\mathbb{B}_s.$$

Here, in the second expression, the formal variable  $t$  marks the symbol one. There is first a formal direct decomposition of the quasi-inverse,

$$(I - v\mathbb{L}_{s,t})^{-1} = (I - v(\mathbb{A}_s + t\mathbb{B}_s))^{-1} = (I - v\mathbb{A}_s)^{-1} \circ (I - t(v\mathbb{B}_s \circ (I - v\mathbb{A}_s)^{-1}))^{-1},$$

that introduces the (weighted) operator

$$(43) \quad \mathbb{G}_{v,s} = v\mathbb{B}_s \circ (I - v\mathbb{A}_s)^{-1},$$

and leads to the writing

$$(44) \quad (I - v\mathbb{L}_{s,t})^{-1} = (I - v\mathbb{A}_s)^{-1} (I - t\mathbb{G}_{v,s})^{-1}.$$

We now introduce another writing of the previous quasi-inverse, using the two costs defined on  $\Sigma^*$ , the length  $|w|$  of  $w$ , and the number  $n(w)$  of ones. As each of these costs is additive<sup>6</sup>, and due to the morphism (40), it comes

$$(45) \quad v^n \mathbb{L}_s^n = \sum_{w \in \Sigma^n} v^{|w|} \mathbb{H}_s \langle w \rangle, \quad v^n \mathbb{L}_{s,t}^n = \sum_{w \in \Sigma^n} v^{|w|} t^{n(w)} \mathbb{H}_s \langle w \rangle,$$

and the quasi-inverse also writes as

$$(46) \quad (I - v\mathbb{L}_{s,t})^{-1} = \sum_{w \in \{0,1\}^*} v^{|w|} t^{n(w)} \mathbb{H}_s \langle w \rangle.$$

**5.3. The generating function  $\Lambda_\mu$  and the secant  $S[\phi]$ .** With costs  $c \in \{m, n\}$  defined in Sections 2.2 and 2.3, we associate the expectation  $\underline{c}_\mu(k)$  as in Eqn (7) and the generating function  $C_\mu(v)$  as in Eqn (8). We look for alternative expressions of the two generating functions  $M_\mu, N_\mu$ . In this way, their dominant behaviour (their behaviour near their dominant singularity) becomes apparent, and, via singularity analysis, we relate the asymptotics of the two sequences  $\underline{m}_\mu(k)$  and  $\underline{n}_\mu(k)$  ( $k \rightarrow \infty$ ) with the dominant behaviours of  $M_\mu$  and  $N_\mu$ .

In order to obtain alternative expressions for functions  $N_\mu$  and  $M_\mu$ , it proves convenient to first deal with the following trivariate generating function

$$(47) \quad \Lambda_\mu(v, t, s) = \sum_{w \in \{0,1\}^*} v^{|w|} t^{n(w)} p_\mu(w)^s,$$

where, as previously, the variable  $v$  marks the length  $|w|$  of the word  $w$  (its number of symbols), and the variable  $t$  marks the number of ones in  $w$ . We remark the easy (and useful) equality,

$$(48) \quad \Lambda_\mu(v, 1, 1) = \sum_{n \geq 0} v^n \sum_{w \in \{0,1\}^n} p_\mu(w) = \sum_{n \geq 0} v^n = \frac{1}{1-v}.$$

Assume that we have obtained, in the case of a source of interest, an alternative expression for  $\Lambda_\mu$ . The two generating functions  $N_\mu, M_\mu$  are indeed (formal) derivatives of the function  $\Lambda_\mu$ ,

$$(49) \quad \left\{ \begin{array}{l} N_\mu(v) = \sum_{w \in \{0,1\}^*} v^{|w|} n(w) p(w) = \frac{\partial}{\partial t} \Lambda_\mu(v, t, s)|_{s=1, t=1}, \\ M_\mu(v) = \sum_{w \in \{0,1\}^*} v^{|w|} |\log p(w)| p(w) = -\frac{\partial}{\partial s} \Lambda_\mu(v, t, s)|_{s=1, t=1}, \end{array} \right\}$$

and the alternative expression for  $\Lambda_\mu$  may be used to obtain expressions for generating functions  $M_\mu$  and  $N_\mu$ .

<sup>6</sup>For two words  $w$  and  $w'$ , one has  $|w \cdot w'| = |w| + |w'|$ , and  $n(w \cdot w') = n(w) + n(w')$

We have already obtained alternative expressions for the quasi-inverse  $(I - \mathbb{L}_{s,t})^{-1}$  in Eqn (44) and (46). We now return to the function  $\Lambda_\mu(v, t, s)$  itself, with the expression (47), and wish to obtain an alternative expression of  $\Lambda_\mu$ .

When  $\mu$  is a measure of density  $\phi$ , this is the role of the function  $S[\phi]$ :

**Definition 24.** [The function  $S[\phi]$ ] *With a function  $\phi$  strictly positive that belongs to  $\mathcal{C}^1([0, 1])$ , one associates the function  $S[\phi]$  defined on  $[0, 1]^2$  by*

$$(50) \quad S[\phi](x, y) = \frac{1}{y-x} \int_x^y \phi(t) dt \quad (x \neq y), \quad S[\phi](x, x) = \phi(x) \quad .$$

Remark that the equality also holds

$$S[\phi](x, y) = \left| \frac{1}{y-x} \int_x^y \phi(t) dt \right|.$$

We now explain how this function  $S[\phi]$  intervenes in the expression of the trivariate function  $\Lambda_\mu$  and how it is involved in Shannon weights. We also describe its main properties:

**Lemma 25.** [Generating operators and Shannon weights.] *Consider a measured dynamical system  $(\mathcal{S}, \mu)$  that satisfies Definition 6, together with its associated transfer operators. The following holds:*

- (a) *The secant operator  $\mathbb{H}_s\langle w \rangle$  is a generating operator for the fundamental intervals  $\mathcal{I}_w$ :*

$$\mu(\mathcal{I}_w)^s = \mathbb{H}_s\langle w \rangle[S^s[\phi]](0, 1).$$

- (b) *There is an alternative expression for the function  $\Lambda_\mu$ ,*

$$\Lambda_\mu(v, t, s) = (I - \mathbb{L}_{s,t})^{-1}[S^s[\phi]](0, 1).$$

- (c) *The Shannon weights of the source  $(\mathcal{S}, \mu)$  are expressed with the transfer operator  $\mathbb{L}_s$  of  $\mathcal{S}$  defined in (41),*

$$\underline{m}_\mu(n) = -\frac{d}{ds} \left( \mathbb{L}_s^n[[S^s[\phi]]] \right) \Big|_{s=1} (0, 1) \quad ;$$

- (d) *When  $\phi$  is strictly positive and belongs to  $\mathcal{C}^1([0, 1])$ , the function  $S[\phi]$  belongs to  $\mathcal{C}^1([0, 1]^2)$  with a norm<sup>7</sup>*

$$\|S^s[\phi]\|_0 \leq r_\phi^s, \quad \|D(S[\phi]^s)\|_0 \leq s r_\phi^{s-1} \|\phi'\|_0,$$

*that involves the constant  $r_\phi = \max(\|\phi\|_0, \|1/\phi\|_0)$ .*

*Proof.* Item (a) With the definition of the function  $S[\phi]$  and operators  $\mathbb{H}_s\langle w \rangle$ , one has, for  $(x, y) = (0, 1)$

$$\begin{aligned} \mathbb{H}_s\langle w \rangle[S^s[\phi]](0, 1) &= |h_w(0) - h_w(1)|^s [S^s[\phi]](h_w(0), h_w(1)) \\ &= |h_w(0) - h_w(1)|^s \left| \frac{1}{h_w(0) - h_w(1)} \right|^s \left| \int_{h_w(0)}^{h_w(1)} \phi(u) du \right|^s = \left| \int_{h_w(0)}^{h_w(1)} \phi(u) du \right|^s = \mu(\mathcal{I}_w)^s. \end{aligned}$$

Item (b) Clear with (46).

Item (c) The  $n$ -th Shannon weight is (by definition) the entropy of the fundamental partition  $\mathcal{P}_n$ . With Eqn (21) and Item (a), one has

$$\sum_{w \in \{0,1\}^n} \mu(\mathcal{I}_w)^s = \sum_{w \in \{0,1\}^n} \mathbb{H}_s\langle w \rangle[S^s[\phi]](0, 1) = \mathbb{L}_s^n[S^s[\phi]](0, 1),$$

and taking the (formal) derivative with respect to  $s$  leads to Item (c).

<sup>7</sup>The norm  $\|\cdot\|_0$  is the norm sup and the norm  $\|F\|_1$  on  $\mathcal{C}^1([0, 1]^2)$  is  $\|F\|_1 = \|F\|_0 + \|DF\|_0$

Item (d) The function  $F$  defined by  $F(x) = \int_0^x \phi(t)dt$  belongs to  $\mathcal{C}^2([0, 1])$  and satisfies  $F'(x) = \phi(x)$ . For  $x \neq y$ , one has

$$D(S[\phi]^s)(x, y) = s S[\phi]^{s-1} D(S[\phi])(x, y),$$

and, with the Mean Value Theorem,

$$\frac{\partial}{\partial x} S[\phi](x, y) = \frac{1}{(y-x)^2} (F(y) - F(x) - (y-x)F'(x)) = \frac{1}{2} F''(t) = \frac{1}{2} \phi'(t)$$

for some  $t \in ]x, y[$ . This entails<sup>8</sup>, for any pair  $(x, y)$  with  $x \neq y$ ,

$$|D(S[\phi])(x, y)| \leq \|\phi'\|_0.$$

At the point  $(x, x)$  of the diagonal, one has, with the Mean Value Theorem, with possibly  $x = 0$  and any  $z$ ,

$$S[\phi](z, x) - S[\phi](x, x) = \frac{1}{(z-x)} (F(z) - F(x) - (z-x)F'(x)) = \frac{1}{2} (z-x)F''(t)$$

with some  $t \in ]z, x[$ . This entails the continuity of  $S[\phi]$  at point  $(x, x)$  together with the bound, for any  $(x, x)$ ,

$$|D(S[\phi])(x, x)| \leq \|\phi'\|_0. \quad \square$$

**5.4. Alternative expression for  $\Lambda_\mu$ .** We give a summary of the results we have obtained in the previous subsections:

**Theorem 26.** [Alternative expressions for a measured dynamical system  $(I, T, \mu)$  of the Class  $\mathcal{DR}$ ] *Consider a dynamical system  $(\mathcal{I}, T)$  of the Class  $\mathcal{DR}$ , and denote by  $\mathbb{L}_s, \mathbb{G}_{v,s}, \mathbb{A}_s$  the various associated transfer operators. The following holds*

(a) *There is a relation between the three (secant) quasi-inverses,*

$$(51) \quad (I - v\mathbb{L}_{s,t})^{-1} = (I - v\mathbb{A}_s)^{-1} \circ (I - t\mathbb{G}_{v,s})^{-1},$$

*and a similar expression for the tangent quasi-inverses.*

(b) *The trivariate generating function  $\Lambda_\mu$  involves the sequence  $q(n) = a^n(1)$ , together with the function  $S[\phi]$ ,*

$$\Lambda_\mu(v, t, s) = (I - v\mathbb{L}_{s,t})^{-1} [S^s[\phi]](0, 1) = \sum_{n \geq 0} v^n q(n)^s (I - t\mathbb{G}_{v,s})^{-1} [S^s[\phi]](0, q(n)).$$

*Proof.* We have already proven the first expression of Item (b). We now focus on the second one: The iterate  $\mathbb{A}_s^n$  satisfies, for a function  $L$  defined on  $\mathcal{I}^2$  and  $(x, y) \in \mathcal{I}^2$ ,

$$\mathbb{A}_s^n[L](x, y) = \left| \frac{a^n(x) - a^n(y)}{x - y} \right|^s L(a^n(x), a^n(y)).$$

The branch  $a^n$  satisfies  $a^n(0) = 0$  and  $a^n(1) = q(n)$ ; thus,  $\mathbb{A}_s^n[L](0, 1)$  together with  $(I - v\mathbb{A}_s)^{-1}[L](0, 1)$  are expressed with the sequence  $q(n)$  as

$$\mathbb{A}_s^n[L](0, 1) = q(n)^s L(0, q(n)), \quad (I - v\mathbb{A}_s)^{-1}[L](0, 1) = \sum_{n \geq 0} v^n q(n)^s L(0, q(n)). \quad \square$$

This expression of  $\Lambda_\mu(s, v, t)$  will be the starting point of our further analysis, done in the sequel of this Section and in the next Section 6.

<sup>8</sup>We use the norm  $|D(G)(x, y)| = \left| \frac{\partial}{\partial x} G(x, y) \right| + \left| \frac{\partial}{\partial y} G(x, y) \right|$

**5.5. Properties of the operators for a source of the Good Class.** Our main final interest is the study of the (weighted) operator  $\mathbb{G}_{v,s}$  defined in (43) for a real  $v \leq 1$  and  $s > s_0$  (where  $s_0$  is the convergence abscissa of the Dirichlet series  $\Delta(s)$ ). The operator  $\mathbb{G}_{v,s}$  is an extension of the (unweighted) secant operator  $\mathbb{G}_{1,s}$ , also denoted as  $\mathbb{G}_s$ , and defined as

$$\mathbb{G}_s[F](x, y) = \sum_{m \geq 1} \left| \frac{g_m(x) - g_m(y)}{x - y} \right|^s F(g_m(x), g_m(y)),$$

in terms of the inverse branches  $g_m = a^{m-1} \circ b$ . With Definition (41), the operator  $\mathbb{G}_s$  is the secant transfer operator of the block dynamical system  $\mathcal{B}$ . When the system  $\mathcal{B}$  belongs to the Good Class (see Definition 17), the operator  $\mathbb{G}_s$  is deeply studied in the paper [CV15]. This paper first relates the secant operator  $\mathbb{G}_s$  to the (classical) tangent operator

$$\mathbf{G}_s[f](x) = \sum_{m \geq 1} |g'_m(x)|^s f(g_m(x));$$

it uses the equality (38) that proves that the tangent operator coincides with the secant operator on the diagonal. The paper [CV15] considers the case when the operators resp. act (for the tangent operator) on the functional space  $\mathcal{C}^1(\mathcal{I})$ , endowed with the norm

$$\|f\| = \|f\|_0 + \|f'\|_0, \quad \|g\|_0 = \sup_{x \in \mathcal{I}} \|g(x)\|,$$

and (for the secant operator) on  $\mathcal{C}^1(\mathcal{I} \times \mathcal{I})$ , endowed with the norm

$$\|F\| = \|F\|_0 + \|DF\|_0, \quad \|G\|_0 = \sup_{(x,y) \in \mathcal{I}^2} \|G(x, y)\|,$$

and relates its spectral properties on these respective spaces.

In its Theorem 5.1, the paper [CV15] deals with the notion of quasi-compactity<sup>9</sup>; it states, for the operators  $\mathbf{G}_s$  and  $\mathbb{G}_s$ , dominant spectral properties and relates the dominant spectral properties of  $\mathbb{G}_s$  to these of  $\mathbf{G}_s$ . We easily extend Theorem 5.1 of [CV15] to the weighted version of the operators for  $0 \leq v \leq 1$ . We also explicitly state in Item (iii) the existence of an invariant density.

**Proposition 27.** *Consider a dynamical system  $\mathcal{B}$  of the Good Class and real pairs  $(v, s)$  with  $0 \leq v \leq 1, s > s_0$ , where  $s_0$  is the convergence abscissa of the Dirichlet series  $\Delta(s)$ . The following holds for the weighted secant operator  $\mathbb{G}_{v,s}$*

- (i) *The operator  $\mathbb{G}_{v,s}$  acts on the space  $\mathcal{C}^1(\mathcal{I}^2)$  and is quasi-compact.*
- (ii) *The decomposition holds*

$$\mathbb{G}_{v,s} = \lambda(v, s) \mathbb{P}_{v,s} + \mathbb{Q}_{v,s},$$

*where  $\lambda(v, s)$  is the dominant eigenvalue of  $\mathbb{G}_{v,s}$ ,  $\mathbb{P}_{v,s}$  is the dominant projector and  $\mathbb{Q}_{v,s}$  is the remainder operator of the operator  $\mathbb{G}_{v,s}$ .*

- (iii) *The two dominant eigenvalues (resp. relative to  $\mathbb{G}_{v,s}$  and  $\mathbf{G}_{v,s}$ ) are equal. At  $(v, s) = (1, 1)$  the dominant eigenvalue  $\lambda(v, s)$  equals 1; the two dominant eigenfunctions, the density  $\psi = \psi_{1,1}$  on  $\mathcal{I}$  and the density  $\Psi = \Psi_{1,1}$  on  $\mathcal{I}^2$  are invariant by  $\mathbf{G}_1$  (resp.  $\mathbb{G}_1$ ) and the measure  $\nu$  of density  $\psi$  is invariant by the block map  $\hat{T}$ . Moreover, the equalities hold*

$$(52) \quad \Psi(0, 1) = 1, \quad \mathbb{P}_{1,1}[F](x, y) = I[F] \Psi(x, y). \quad \text{with} \quad I[F] = \int_{\mathcal{I}} F(t, t) dt.$$

<sup>9</sup>An operator is quasi-compact if the upper part of its spectrum is only composed with eigenvalues of finite multiplicity.

*Proof.* Items (i), (ii) and (iii) are easy extensions of their analogs in Theorem 5.1 of the paper [CV15]: Using Lasota Yorke inequalities that hold for operator  $\mathbb{G}_{v,s}$  when  $v \leq 1$  and  $s > \rho$  entails Hennion's Property and quasi-compactity.  $\square$

**5.6. Entropy of the block source.** We give in the next proposition a detailed proof of the properties of the entropy of the source  $\mathcal{B}$  that were not made completely precise in [CV15]. Together, the two propositions 27 and 28 provide a proof for Assertion (a) of Theorem 19.

**Proposition 28.** [Entropy of the block source] *The source  $\mathcal{B}$  admits an entropy with respect to the invariant measure  $\nu$  that satisfies*

$$\mathcal{E}_\nu(\mathcal{B}) = -\lambda'(1) = \int_0^1 \widehat{\mathbf{G}}_1[\psi(t)] dt < \infty, \quad \widehat{\mathbf{G}}_1 = \frac{d}{ds} \mathbf{G}_s|_{s=1}.$$

*Proof.* We begin with the expression of Shannon weights (of the source  $\mathcal{B}$ ) given in Lemma 25 (b). It involves the transfer operator  $\mathbb{G}_s$  of the system  $\mathcal{B}$ , and more precisely the derivatives  $s \mapsto (d/ds)(\mathbb{G}_s^n)$  of its iterates at  $s = 1$ . As  $\mathcal{B}$  is of Good Class, its transfer operator  $\mathbb{G}_s$  has, on a real neighborhood of  $s = 1$ , a dominant eigenvalue  $\lambda(s)$  with  $\lambda(1) = 1$  together with a spectral gap. Then, with Eqn (52) and Lemma 25(a), the decomposition holds on this neighborhood,

$$\mathbb{G}_s^n[S^s[\phi]](0, 1) = \lambda(s)^n \mathbb{P}_{1,1}[S[\phi]](0, 1) (1 + O(\rho^n)) = \lambda(s)^n (1 + O(\rho^n)),$$

where the  $O$  is uniform on this neighborhood. Then, using Lemma 25(c) and taking the derivative at  $s = 1$  leads to the estimate of Shannon weights of the source  $\mathcal{B}$ ,

$$\underline{m}_\mu(n) = -n\lambda'(1) (1 + O(\rho^n)).$$

Then, for any measure  $\mu$  with a strictly positive density of Class  $\mathcal{C}^1$ , the entropy rate  $\mathcal{E}_\mu$  of the block source exists, does not depend on  $\mu$  and coincides with  $-\lambda'(1)$ .

We now apply Proposition 11 in this context. The system  $\mathcal{B}$  is generating, and with Proposition 27 Item(iii) it admits an invariant measure (namely, the measure  $\nu$  of density  $\psi$ ). Then, with Proposition 11, the entropy rate  $\mathcal{E}_\nu = -\lambda'(1)$  coincides with the metrical entropy  $\mathcal{E}_\nu(\mathcal{B})$  of the block dynamical system  $\mathcal{B}$ .

We now prove the equality

$$(53) \quad \lambda'(1) = \int_{\mathcal{I}} \widehat{\mathbf{G}}_1[\psi](t) dt \quad \psi \text{ the invariant density of } \mathcal{B},$$

that is known as the Rohlin Formula. We begin with the perturbative equation

$$\mathbf{G}_s[\psi_s] = \lambda(s) \psi_s$$

relative to the eigenvalue  $\lambda(s)$ , together with its eigenvector  $\psi_s$  with the normalisation  $\int_{\mathcal{I}} \psi_s(t) dt = 1$ . We have  $\psi_1 = \psi$  and  $\lambda(1) = 1$ . We consider its derivative with respect to  $s$  at  $s = 1$  and take the integral on  $\mathcal{I}$ . We first obtain, two terms, resp. equal to

$$\int_0^1 \left( \frac{d}{ds} \mathbf{G}_s|_{s=1} \right) [\psi](t) dt \quad \text{and} \quad \left( \frac{d}{ds} \lambda(s)|_{s=1} \right) \int_0^1 \psi(t) dt = \lambda'(1).$$

There are of course two other terms in the derivative at  $s = 1$ , one for the left member and one for the right member, resp. equal to

$$\int_0^1 \mathbf{G}_1[\underline{\psi}](t) dt \quad \text{and} \quad \int_0^1 \underline{\psi}(t) dt, \quad \text{with } \underline{\psi} := (d/ds) \psi_s|_{s=1},$$

and indeed equal, as the operator  $\mathbf{G}_1$  is a density transformer.

There is another expression (more classical) for the Rohlin formula, when applied to a system  $\mathcal{B}$  of the Good Class associated with a shift  $\widehat{T}$ ,

$$\mathcal{E}_\nu(\mathcal{B}) = \int_{\mathcal{I}} \log |\widehat{T}'(x)| \psi(x) dx,$$

where  $\psi$  is the invariant density of the operator  $\mathbf{G}_1$ . The previous expression coincides with (53) due to the change of variables  $x = g_m(u)$  on each interval  $\mathcal{J}_m$ .  $\square$

**5.7. Invariant measure for a source in the Class  $\mathcal{DRBGC}$ .** We now wish to prove Assertion (b) of Theorem 19. We mainly use (tangent) transfer operators (all taken at  $s = 1$ ) defined in Section 5.1, we omit the index  $s$ , and we notably deal with the operators  $\mathbf{L}, \mathbf{A}, \mathbf{B}, \mathbf{G}$ .

We recall the decomposition obtained in Section 5.1

$$(54) \quad (I - \mathbf{L})^{-1} = (I - \mathbf{A})^{-1} \circ (I - \mathbf{G})^{-1}, \quad \mathbf{G} = \mathbf{B} \circ (I - \mathbf{A})^{-1}.$$

**Proposition 29.** *Consider a source  $\mathcal{S}$  of the  $\mathcal{DRBGC}$  Class, whose block source  $\mathcal{B}$  admits the invariant density  $\psi$  and the invariant probability  $\nu$ . Then,*

- (i) *the function  $\phi_0 = (I - \mathbf{A})^{-1}[\psi]$  is invariant by the operator  $\mathbf{L}$ . Its associated measure has a total mass equal to  $\mathbb{E}_\nu[W]$ . When the mean  $\mathbb{E}_\nu[W]$  is finite, this leads to an invariant probability measure  $\pi$  with density  $\rho = (1/\mathbb{E}_\nu[W])\phi_0$ , which satisfies  $\lim_{x \rightarrow 0} \rho(x) = +\infty$ .*
- (ii) *The following equalities relate the probability measure  $\nu$  of density  $\psi$  and the probability measure  $\pi$  associated with  $\rho$ , for any  $k \geq 0$ ,*

$$\nu[T^{-k}\mathcal{J}] = \pi[\mathcal{J} \cap T^{-k-1}(\mathcal{J})], \quad \mathcal{J} = b(\mathcal{I}).$$

There is a proof for Item (i) in [S95], pages 120-121 (Theorem 17.1.3). Dealing here with the fundamental decomposition described in (54) and densities, we now describe a direct and simple proof of these results, which moreover uses easy relations between various sets collected in the following lemma (whose proof is omitted):

**Lemma 30.** *Consider a source  $\mathcal{S}$  of the  $\mathcal{DR}$  Class, with  $\mathcal{J} = b(\mathcal{I})$  and the waiting time  $W$  of symbol one. For  $X \subset \mathcal{I}$ , there are easy relations :*

- (a)  $T^{-k}\mathcal{J} \cap [W > k] = [W = k + 1] \quad k \geq 0$
- (b)  $a^n(X) = T^{-n}X \cap [W > n]$
- (c)  $b(X) = \mathcal{J} \cap T^{-1}(X)$

*Proof of Proposition 29 .*

Item (i). With Relation (54), the function  $\rho = (I - \mathbf{A})^{-1}[\psi]$  satisfies

$$(I - \mathbf{L})[\rho] = (I - \mathbf{G})(I - \mathbf{A})[\rho] = (I - \mathbf{G})[\psi] = 0$$

and it is thus invariant by  $\mathbf{L}$ . For any  $y \in [0, 1]$ , one has

$$\rho(y) = \sum_{n=0}^{\infty} \mathbf{A}^n[\psi](y) = \sum_{n=0}^{\infty} (a^n)'(y)\psi(a^n(y)).$$

As the system  $(I, T)$  belongs to the class  $\mathcal{DR}$  defined in Definition 12, the branch  $a$  has an indifferent point at 0 i.e.,  $a(0) = 0, a'(0) = 1$ , and, as already mentioned in Section 3.3, the derivatives  $(a^n)'(0)$  are all equal to 1, and  $\rho(0) = +\infty$ . This implies that  $\rho$  cannot be of class  $\mathcal{C}^1$  on  $\mathcal{I}$ .

Moreover, for any  $X \subset \mathcal{I}$ , the equalities hold

$$\int_X \rho(y)dy = \sum_{n=0}^{\infty} \int_X \mathbf{A}^n[\psi](y) dy = \sum_{n=0}^{\infty} \int_{a^n(X)} \psi(t) dt = \sum_{n=0}^{\infty} \nu(a^n(X)).$$

Furthermore, Lemma 30(b) with  $X = \mathcal{I}$  entails the equalities

$$\int_{\mathcal{I}} \rho(y)dy = \sum_{n=0}^{\infty} \nu[W > n] = \mathbb{E}_\nu[W],$$

and, with Lemma 30(a), the relation  $\pi(\mathcal{J}) = 1 - \nu[W = 0] = 1$  holds.

Item (ii) With Lemma 30(c), for any  $k \geq 0$  and  $Y = T^{-k}\mathcal{J}$ , the relation holds

$$b(Y) = \mathcal{J} \cap T^{-(k+1)}(Y).$$

Moreover the relation  $\mathbf{B}[\rho] = \psi$  entails the equality  $\nu(Y) = \pi(b(Y))$ , and thus (iii).  $\square$

This Section has provided a proof for Theorem 19. We now deal, in the next Section, with the proof of our second main result, Theorem 20.

## 6. DYNAMICAL ANALYSIS: PROOF OF THE SECOND MAIN RESULT.

This Section provides a proof for Theorem 20. Theorem 26 leads to study in Section 6.1 the behaviour of the quasi-inverse  $v \mapsto (I - \mathbb{G}_{v,1})^{-1}$  as  $v \rightarrow 1^-$ . Via Perturbation theory, it is related later in Proposition 32 of Section 6.2 to the behaviour of  $v \mapsto 1 - \lambda(v, 1)$  (as  $v \rightarrow 1^-$ ). With these two estimates, we study, as  $v \rightarrow 1^-$ , the generating functions  $N_\mu$  and  $M_\mu$ . This leads to the final part of the proof of Theorem 20.

### 6.1. Perturbation of the transfer operator as $v \rightarrow 1^-$ .

**Proposition 31.** [Perturbation as  $v \rightarrow 1^-$ ] *The notations are the same as in Proposition 27. For a function  $F$  of class  $\mathcal{C}^1(\mathcal{I}^2)$ , we denote*

$$I[F] = \int_{\mathcal{I}} F(u, u) du.$$

When  $I[F]$  is non zero, the following estimate holds for the quasi-inverse,

$$(I - \mathbb{G}_{v,1})^{-1}[F](x, y) = \frac{I[F]}{1 - \lambda(v, 1)} \Psi(x, y) (1 + \delta(v) \|F\|), \quad \delta(v) \rightarrow 0 \quad (v \rightarrow 1^-).$$

*Proof.* For a real  $v$  with  $0 \leq v < 1$ , using the relations

$$\mathbb{P}_{v,1}^2 = \mathbb{P}_{v,1}, \quad \mathbb{P}_{v,1} \circ \mathbb{Q}_{v,1} = \mathbb{Q}_{v,1} \circ \mathbb{P}_{v,1} = 0,$$

the spectral decomposition of  $\mathbb{G}_{v,1}$  described in Proposition 27 (ii) leads to the decomposition of the quasi-inverse, for  $F \in \mathcal{C}^1(\mathcal{I}^2)$ ,

$$(55) \quad (I - \mathbb{G}_{v,1})^{-1}[F] = \frac{\lambda(v, 1)}{1 - \lambda(v, 1)} \mathbb{P}_{v,1}[F] + \mathbb{R}_{v,1}[F], \quad \mathbb{R}_{v,1} = (I - \mathbb{Q}_{v,1})^{-1}.$$

We now let  $v \rightarrow 1^-$  and use a version of Abel's Continuity Theorem on Banach spaces.

*Abel's Continuity Theorem on Banach spaces.* On a Banach space  $\mathcal{A}$ , consider a series  $G(v) = \sum_{m \geq 1} v^m G_m$  which involves a sequence  $G_m \in \mathcal{A}$  and assume that the series  $G(v)$  is convergent in  $\mathcal{A}$  for  $v$  real in  $[0, 1]$ . Then  $\lim_{v \rightarrow 1^-} G(v) = G(1)$ .

We apply the previous result to the operator  $G(v) = \mathbb{G}_{v,1}$  on the Banach space of the operators which act on  $\mathcal{C}^1(\mathcal{I}^2)$ . This proves that the operator  $\mathbb{G}_{v,1}$  tends to  $\mathbb{G}_{1,1}$  as  $v \rightarrow 1^-$ .

Via the theory of continuous perturbation, [see for instance [K80] Theorem 3.16, page 212], the three spectral elements  $\lambda(v, 1)$ ,  $\mathbb{P}_{v,1}$  and  $\mathbb{Q}_{v,1}$  of  $\mathbb{G}_{v,1}$  satisfy the following, as  $v \rightarrow 1^-$

$$\lambda(v, 1) \text{ tends to } \lambda(1, 1) = 1, \quad \mathbb{P}_{v,1} \text{ tends to } \mathbb{P}_{1,1} \text{ and } \mathbb{Q}_{v,1} \text{ tends to } \mathbb{Q}_{1,1},$$

and thus 
$$\delta(v) = \sup \left[ (\lambda(v, 1) - 1), \|\mathbb{P}_{v,1} - \mathbb{P}_{1,1}\|, \|\mathbb{Q}_{v,1} - \mathbb{Q}_{1,1}\| \right] \rightarrow 0.$$

Due to the spectral gap, the operator  $(1 - \mathbb{Q}_{1,1})^{-1}$  has a norm at most  $1/(1 - \rho)$  with  $\rho < 1$ . If we choose  $v$  close enough to 1, so that the operator  $\mathbb{Q}_{v,1} - \mathbb{Q}_{1,1}$  has a norm  $\delta(v)$  with  $\delta(v) < (1 - \rho)$  then, the decomposition

$$\mathbb{R}_{v,1} - \mathbb{R}_{1,1} = (I - \mathbb{Q}_{1,1})^{-1} \left[ (I + (\mathbb{Q}_{v,1} - \mathbb{Q}_{1,1})(I - \mathbb{Q}_{1,1})^{-1})^{-1} - I \right]$$

entails the bound  $\|\mathbb{R}_{v,1} - \mathbb{R}_{1,1}\| = O(\delta(v))$ .

Finally, using Eqn (52), the equality  $\mathbb{P}_{1,1}[F](x, y) = I[F]\Psi(x, y)$  holds, and the decomposition given in (55) entails the estimate

$$(56) \quad (I - \mathbb{G}_{v,1})^{-1}[F](x, y) = \frac{\lambda(v, 1)}{1 - \lambda(v, 1)} I[F] \Psi(x, y) + \frac{\delta(v)}{1 - \lambda(v, 1)} \epsilon(v, F)$$

$$\text{with } \epsilon(v, F) = \|F\| (1 + \|\mathbb{R}_{1,1}\| + O(\delta(v))) = O(\|F\|).$$

This ends the proof.  $\square$

**6.2. Behaviour of  $1 - \lambda(v, 1)$  as  $v \rightarrow 1^-$ .** The previous result leads to study the dominant behaviour of  $1 - \lambda(v, 1)$  as  $v \rightarrow 1^-$ . We relate this dominant behaviour to the dominant behaviour of  $Q_\nu(v)$  as  $v \rightarrow 1^-$ .

**Proposition 32.** *The ratio*

$$(57) \quad D(v) = \frac{1 - \lambda(v, 1)}{1 - v} \quad \text{satisfies} \quad D(v) \sim_{v \rightarrow 1^-} Q_\nu(v) \quad (v \rightarrow 1^-),$$

and involves the generating function  $Q_\nu(v)$  of the wtd sequence (wrt to the invariant measure  $\nu$  of the block source). When  $Q_\nu(1) < \infty$ , the map  $v \mapsto \lambda(v, 1)$  is derivable on the left at  $v = 1$ , with a derivative equal to  $Q_\nu(1)$ .

*Proof.* We begin with the expression of Theorem 26, at  $(t, s) = (1, 1)$ , with a measure  $\mu$  of density  $\phi$

$$\Lambda_\mu(v, 1, 1) = \sum_{n \geq 0} v^n q(n) (I - \mathbb{G}_{v,1})^{-1} [S[\phi]](0, q(n)).$$

Proposition 31 describes the dominant part of the quasi-inverse  $(I - \mathbb{G}_{v,1})^{-1}$  as  $v \rightarrow 1^-$  and leads to the following estimate,

$$(58) \quad \text{Dom } ({}_{1^-})\Lambda_\mu(v, 1, 1) = \frac{\tilde{D}(v)}{1 - \lambda(v, 1)}, \quad \tilde{D}(v) = \sum_{n \geq 0} v^n q(n) \Psi(0, q(n)).$$

It involves the invariant density  $\Psi$  of  $\mathbb{G}_1$  that extends the invariant density  $\psi$  of  $\mathbf{G}_1$ . On the other hand, the equality described in Eqn (48) introduces  $D(v)$  defined in (57) and proves with (58) the estimate

$$(59) \quad D(v) \sim_{v \rightarrow 1^-} \tilde{D}(v).$$

As the function  $\Psi$  belongs to  $\mathcal{C}^1(\mathcal{I}^2)$  and the sequence  $q(n)$  tends to zero, the sequence  $\Psi(0, q(n))$  satisfies

$$\Psi(0, q(n)) = \Psi(0, 0) + O(q(n)),$$

and, as  $\Psi(0, 0) = \psi(0)$  is strictly positive, the estimate holds

$$(60) \quad \tilde{D}(v) = \psi(0) Q_\tau(v) + O(R(v)) \quad R(v) = \sum_{n \geq 0} q(n)^2 v^n,$$

and involves the generating function  $Q_\tau(v)$  of the wtd sequence with respect to the uniform measure  $\tau$ .

There are now two cases for the estimate of  $\tilde{D}(v)$  as  $v \rightarrow 1^-$  obtained in Eqn (60), and we prove in each case the estimate  $\tilde{D}(v) \sim_{v \rightarrow 1^-} Q_\nu(v)$ .

*First Case:*  $Q_\tau(v) \rightarrow \infty$  as  $v \rightarrow 1^-$ . As  $q(n) \rightarrow 0$ , the sequence  $q(n)^2$  is  $o(q(n))$ . Then Lemma 34 Item (i) applies and the series  $R(v)$  in (60) is  $o(Q_\tau(v))$  as  $v \rightarrow 1^-$ . This entails, with (60),

$$(61) \quad \tilde{D}(v) \sim_{v \rightarrow 1^-} \psi(0) Q_\tau(v).$$

Furthermore, Lemma 34 Item (ii)(b) entails the estimate  $Q_\nu(v) \sim_{v \rightarrow 1^-} \psi(0) Q_\tau(v)$ , and the estimate  $\tilde{D}(v) \sim_{v \rightarrow 1^-} Q_\nu(v)$  holds.

*Second Case:*  $Q_\tau(1) < \infty$ . Then (60) entails that  $\tilde{D}(1) < \infty$  too, and, with (59), the equality  $D(1) = \tilde{D}(1)$  holds. We now prove the estimate

$$\tilde{D}(1) = Q_\nu(1) = \mathbb{E}_\nu[W].$$

In this case, the map  $v \mapsto \lambda(v, 1)$  is derivable on the left at  $v = 1$ . We denote by  $\psi_{v,1}$  the eigenfunction of  $\mathbf{G}_{v,1}$  relative to the eigenvalue  $\lambda(v, 1)$ , with the normalisation  $\int_{\mathcal{I}} \psi_{v,1}(t) dt = 1$ ; we consider the perturbative equation

$$(62) \quad \mathbf{G}_{v,1}[\psi_{v,1}] = \lambda(v, 1)\psi_{v,1},$$

and its derivative with respect to  $v$  for  $v < 1$ ; we take the integral on  $\mathcal{I}$ , and finally we let  $v = 1$ . We first obtain, at  $v = 1$ , two terms, resp. equal to

$$(63) \quad \int_0^1 \left( \frac{d}{dv} \mathbf{G}_{v,1} \Big|_{v=1} \right) [\psi](t) dt \quad \text{and} \quad \left( \frac{d}{dv} \lambda(v, 1) \Big|_{v=1} \right) \int_0^1 \psi(t) dt = D(1).$$

One has

$$\int_0^1 \mathbf{G}_{v,1}[\psi](t) dt = \sum_{m \geq 1} v^m \int_0^1 |g'_m(t)| \psi \circ g_m(t) dt = \sum_{m \geq 1} v^m \int_{\mathcal{J}_m} \psi(u) du,$$

with  $\mathcal{J}_m = [W = m]$ . Then the first term in (63) is

$$\int_0^1 \left( \frac{d}{dv} \mathbf{G}_{v,1} \Big|_{v=1} \right) [\psi](t) dt = \sum_{m \geq 1} m \nu[W = m] = \mathbb{E}_\nu[W].$$

There are of course two other terms in the derivative at  $v = 1$  of Eqn (62), one for the left member and one for the right member, resp. equal to

$$\int_0^1 \mathbf{G}_1[\psi](t) dt \quad \int_0^1 \psi(t) dt, \quad \text{with } \psi := (d/dv) \psi_{v,1} \Big|_{v=1},$$

and indeed equal, as the operator  $\mathbf{G}_1$  is a density transformer.  $\square$

In the previous two Sections, along the proof of Propositions 31 and 32, we have shown the two estimates as  $v \rightarrow 1^-$

$$(64) \quad (I - \mathbb{G}_{v,1})^{-1}[F](x, y) \sim I[F] \frac{\Psi(x, y)}{(1-v)Q_\nu(v)}, \quad \sum_n q(n)v^n \Psi(0, q(n)) \sim Q_\nu(v).$$

They involve the gf  $Q_\nu(v)$  of the wtd sequence wrt to the invariant measure  $\nu$  of the block source, and the invariant density  $\Psi$  of the operator  $\mathbb{G}_1$  with  $\Psi(0, 1) = 1$ .

**6.3. Study of the generating functions  $N_\mu$  and  $M_\mu$  as  $v \rightarrow 1^-$ .** Starting with the expression of  $\Lambda_\mu$  obtained in Theorem 26,

$$\Lambda_\mu(v, t, s) = \sum_{n \geq 0} q(n)^s v^n (I - t\mathbb{G}_{v,s})^{-1}[S[\phi]^s](0, q(n)),$$

the expressions of the two generating functions  $N_\mu$  and  $M_\mu$  are obtained with derivatives, as already mentioned in Eqn (49): for  $N_\mu$ , the derivative of  $\Lambda_\mu(v, t, 1)$  with respect to  $t$  (at  $t = 1$ ); for  $M_\mu$ , the derivative of  $\Lambda_\mu(v, t, s)$  (with respect to  $s$ , at  $(t, s) = (1, 1)$ ). This leads to the expressions

$$(65) \quad N_\mu(v) = \sum_{n \geq 0} q(n) v^n (I - \mathbb{G}_{v,1})^{-2} \circ \mathbb{G}_{v,1}[S[\phi]](0, q(n)),$$

$$(66) \quad M_\mu(v) = M_\mu^{[0]}(v) + M_\mu^{[1]}(v) + M_\mu^{[2]}(v), \quad \text{with}$$

$$\left\{ \begin{array}{l} M_\mu^{[0]}(v) = \sum_{n \geq 0} q(n) v^n (I - \mathbb{G}_{v,1})^{-1}[S[\phi] \log S[\phi]](0, q(n)), \\ M_\mu^{[1]}(v) = \sum_{n \geq 0} q(n) \log q(n) v^n (I - \mathbb{G}_{v,1})^{-1}[S[\phi]](0, q(n)), \\ M_\mu^{[2]}(v) = \sum_{n \geq 0} q(n) v^n (I - \mathbb{G}_{v,1})^{-1} \circ \widehat{\mathbb{G}}_{v,1} \circ (I - \mathbb{G}_{v,1})^{-1}[S[\phi]](0, q(n)). \end{array} \right.$$

Here,  $\widehat{\mathbb{G}}_{v,1}$  stands for the operator  $(\partial/\partial s)\mathbb{G}_{v,s}|_{s=1}$ ,  $\phi$  is the initial density, and the operator  $S$  is defined in (50).

We first explain the general principles of the analysis: Any of the four expressions above, in (65) or in (66), presents the same structure, with two possible cases (I) or (II),

$$\sum_{n \geq 0} q(n) v^n \mathbb{T}_v[F](0, q(n)) \quad (I) \quad \text{or} \quad \sum_{n \geq 0} q(n) \log q(n) v^n \mathbb{T}_v[F](0, q(n)) \quad (II),$$

and involves an operator  $\mathbb{T}_v$  and a function  $F$ . We first perform an ‘‘inner’’ step and replace the operator  $\mathbb{T}_v$  by its dominant spectral part as  $v \rightarrow 1^-$ : using here the first estimate of Eqn (64), we obtain an estimate of the form

$$\mathbb{T}_v[F](0, q(n)) \sim \frac{I[F]}{\tau(v)} \Psi(0, q(n)),$$

which associates with the operator  $\mathbb{T}_v$  a function  $\tau(v)$  which will be described later, in each case of interest. Collecting all these spectral estimates together, we obtain, for each expression in (65) or in (66), a global estimate of the form

$$\frac{I[F]}{\tau(v)} \left( \sum_{n \geq 0} q(n) v^n \Psi(0, q(n)) \right) \quad (I) \quad \text{or} \quad \frac{I[F]}{\tau(v)} \left( \sum_{n \geq 0} q(n) \log q(n) v^n \Psi(0, q(n)) \right) \quad (II).$$

In each case, this global estimate satisfies

$$(67) \quad \frac{I[F]}{\tau(v)} Q_\nu(v) \quad (I), \quad \text{or} \quad O\left(\frac{H(v)}{\tau(v)}\right) \quad \text{with} \quad H(v) = \sum_n q(n) \log q(n) v^n \quad (II).$$

We now make the computations precise, and derive a precise expression of the term  $\tau(v)$  associated with the operator  $\mathbb{T}_v$ . We obtain

**Proposition 33.** *Consider a dynamical system of the Class DRBGC. The following estimates hold for the generating functions  $N_\mu(v)$  and  $M_\mu(v)$  and involve the generating function  $Q_\nu(v)$  of the wtd sequence (wrt to the invariant measure  $\nu$  of the block source),*

$$(68) \quad (1-v)N_\mu(v) \sim_{v \rightarrow 1^-} \frac{1}{(1-v)Q_\nu(v)};$$

$$(69) \quad (1-v)M_\mu(v) \sim_{v \rightarrow 1^-} I[S[\phi] \log S[\phi]] + O\left(\frac{H(v)}{Q_\nu(v)}\right) + \frac{\mathcal{E}(\mathcal{B})}{(1-v)Q_\nu(v)}.$$

*Proof.* We study each of the four terms of interest.

*Term  $N_\mu$ .* The operator  $\mathbb{T}_v$  equals  $(I - \mathbb{G}_{v,1})^{-2} \circ \mathbb{G}_{v,1}$  and it is applied to  $F = S[\phi]$ . Then, with Eqn (64),

$$\text{Dom}_{(1-)}(I - \mathbb{G}_{v,1})^{-2} \circ \mathbb{G}_{v,1}[S[\phi]] = \frac{\Psi}{(1-v)^2 Q_\nu(v)^2},$$

and  $\tau(v) = (1-v)^2 Q_\nu(v)^2$ . With (67), this leads to the estimate of  $(1-v)N_\mu(v)$  given in (68).

*Terms  $M_\mu^{[0]}(v)$  and  $M_\mu^{[1]}(v)$ .* For the two terms  $M_\mu^{[0]}(v)$  and  $M_\mu^{[1]}(v)$ , the operator  $\mathbb{T}_v$  equals  $(I - \mathbb{G}_{v,1})^{-1}$  and the function  $\tau(v)$  equals  $(1-v)Q_\nu(v)$ . Then, with Eqn (64) and Eqn (67),

$$\text{Dom}_{(1-)}(1-v)M_\mu^{[0]}(v) = I[S[\phi] \log S[\phi]].$$

For the term  $M_\mu^{[1]}(v)$ , we are in case (II), and with Eqn (67), we obtain

$$(70) \quad \text{Dom}_{(1-)}(1-v)M_\mu^{[1]}(v) = O\left(\frac{H(v)}{Q_\nu(v)}\right), \quad H(v) = \sum_{n \geq 0} v^n q(n) \log q(n).$$

Term  $M_\mu^{[2]}(v)$ . The term  $M_\mu^{[2]}(v)$  deals with the operator  $\mathbb{T}_v$  which contains a double quasi-inverse,

$$(71) \quad (I - \mathbb{G}_{v,1})^{-1} \circ \widehat{\mathbb{G}}_{v,1} \circ (I - \mathbb{G}_{v,1})^{-1}$$

and we use two times the spectral decomposition  $(I - \mathbb{G}_{v,1})^{-1}$  given in Proposition 31. With the decomposition of the operator  $\widehat{\mathbb{G}}_{v,1}$  “in the middle”, as

$$(72) \quad \widehat{\mathbb{G}}_{v,1} = a(v)\mathbb{P}_{v,1} + b(v)\mathbb{Q}_{v,1}, \quad \text{and thus} \quad \widehat{\mathbb{G}}_{v,1} \circ \mathbb{P}_{v,1} = a(v)\mathbb{P}_{v,1},$$

the function  $a(v)$  has thus a limit  $a(1)$  as  $v \rightarrow 1^-$  that satisfies, with Proposition 27 (iv),

$$a(1) = a(1)\mathbb{P}_{1,1}[\Psi] = \mathbb{P}_{1,1} \circ \widehat{\mathbb{G}}_{1,1}[\Psi] = \int_0^1 \widehat{\mathbb{G}}_1[\psi](t)dt = \mathcal{E}_\nu(\mathcal{B}).$$

Then, the dominant term associated with the operator in (71) satisfies with (64)

$$\text{Dom}_{(1^-)} \left[ (I - \mathbb{G}_{v,1})^{-1} \circ \widehat{\mathbb{G}}_{v,1} \circ (I - \mathbb{G}_{v,1})^{-1} \right] [S[\phi]] = \frac{\mathcal{E}_\nu(\mathcal{B})}{(1-v)^2 Q_\nu(v)^2} \Psi$$

$$\text{and} \quad \text{Dom}_{(1^-)} \left[ (1-v)M_\mu^{[2]}(v) \right] = \frac{\mathcal{E}_\nu(\mathcal{B})}{(1-v)Q_\nu(v)}.$$

The third estimate of Proposition 33 holds. □

**6.4. End of the proof of the second main result.** For  $(1-v)N_\mu$ , Proposition 33 provides the same estimates as in Assertion (A) of Theorem 20. For  $(1-v)M_\mu(v)$ , Proposition 33 provides estimates that are different from Assertion (A) of Theorem 20 : in particular, the first two terms in (69) do not appear in the estimate of Theorem 20.

The following lemma proves that the estimates obtained in Proposition 33 and in Assertion (A) of Theorem 20 coincide. It also proves Assertion (B) of Theorem 20.

**Lemma 34.** (i) Consider two sequences  $a(n), b(n) > 0$  and the associated generating functions  $A(v)$  and  $B(v)$ . Assume that the sequences  $a(n)$  and  $b(n)$  satisfy

$$a(n) = o(b(n)), \quad \sum_{n=0}^{\infty} b(n) = \infty;$$

then their gf's satisfy  $A(v) = o(B(v))$  as  $v \rightarrow 1^-$ .

(ii) Consider a measured dynamical system  $(\mathcal{S}, \mu)$  in the Class  $\mathcal{DRBGC}$ , the gf's  $H(v)$  defined in (70) and the gf  $Q_\nu(v)$ . The following holds,

(a) The functions  $Q_\nu(v)$  and  $H(v)$  satisfy

$$Q_\nu(v) = o\left(\frac{1}{1-v}\right), \quad H(v) = o\left(\frac{1}{1-v}\right), \quad (v \rightarrow 1^-)$$

The two first terms in (69) are negligible wrt to the last term.

(b) When  $Q_\tau(v) \rightarrow \infty$  as  $v \rightarrow 1^-$ , the estimate holds, for any generating function  $Q_\mu(v)$  relative to a measure  $\mu$  with density  $\phi$

$$Q_\mu(v) - \phi(0)Q_\tau(v) = o(Q_\tau(v)), \quad (v \rightarrow 1^-).$$

(c) The quotient  $D_\mu$  associated in (26) with any measure  $\mu$  is finite.

*Proof.*

Item (i). Fix  $\varepsilon > 0$ . Since  $a(n) = o(b(n))$ , there is  $K = K_\varepsilon > 0$  such that, for  $n \geq K$ , the inequality  $a(n) \leq \varepsilon b(n)$  holds. Thus the generating functions satisfy

$$0 \leq A(v) = \sum_{k=1}^{K-1} a(k)v^k + \sum_{k \geq K} a(k)v^k \leq \sum_{k=1}^{K-1} a(k)v^k + \varepsilon \sum_{k \geq K} b(k)v^k.$$

As  $B(v)$  is divergent as  $v \rightarrow 1^-$ , we remark that the second sum is  $\sim B(v)$  as  $v \rightarrow 1^-$ . Hence, dividing through by  $B(v)$ , the divergence of  $B(v)$  as  $v \rightarrow 1^-$  implies

$$0 \leq \limsup_{v \rightarrow 1^-} A(v)/B(v) \leq \varepsilon.$$

Being  $\varepsilon > 0$  arbitrary, we have proved that  $A(v)/B(v) \rightarrow 0$ .

Item (ii)(a) We use Item (i) with  $b(n) = 1$ , and the two sequences  $q_\nu(n)$  and  $a(n) = -q(n) \log q(n)$ . As the sequences  $q_\nu(n), q(n)$  tend to 0 as  $n \rightarrow \infty$ , the sequences  $q_\nu(n)$  and  $a(n)$  tend to 0, and the two estimates hold.

The two first terms in (69) satisfy

$$I[S[\phi] \log S[\phi]] = O(1) = o\left(\frac{1}{(1-v)Q_\nu(v)}\right), \quad \frac{H(v)}{Q_\nu(v)} = o\left(\frac{1}{(1-v)Q_\nu(v)}\right)$$

and are negligible wrt to the third term.

Item (ii)(b). As the set  $[W > n]$  coincides with the interval  $[0, q_\tau(n)]$ , one has

$$q_\nu(n) = \nu[W > n] = \int_{[W > n]} \psi d\tau = \int_0^{q_\tau(n)} \psi(x) dx.$$

As  $\psi$  is of class  $\mathcal{C}^1$ , one has  $\psi(x) = \psi(0) + O(x)$  as  $x \rightarrow 0$ . As  $q(n) = q_\tau(n)$  tends to 0, we obtain

$$q_\nu(n) = \int_0^{q_\tau(n)} \psi(x) dx = \psi(0) q(n) + O(q(n)^2).$$

We apply Item(i) to  $b(n) = q(n)$  and  $a(n) = q(n)^2 = o(q(n))$ .

Item (ii)(c) In the case when  $Q_\tau(v) \rightarrow \infty$  as  $v \rightarrow 1^-$ , the quotient  $D_\mu$  equals the ratio  $\phi(0)/\psi(0)$  and is thus finite. Otherwise, when  $Q_\tau(1) < \infty$ , the sequences  $q_\mu(n)$  and  $q_\nu(n)$  satisfy

$$q_\mu(n) = \int_0^{q_\tau(n)} \phi(t) dt, \quad q_\nu(n) = \int_0^{q_\tau(n)} \psi(t) dt.$$

As the two densities  $\phi$  and  $\psi$  are  $> 0$  and belong to  $\mathcal{C}^1$ , the associated series are thus convergent with sums resp. equal to  $Q_\mu(1)$  and  $Q_\nu(1)$  and the quotient  $D_\mu$  equals the ratio  $Q_\mu(1)/Q_\nu(1)$ .  $\square$

Consider a measured dynamical system  $(\mathcal{S}, \mu)$  in the Class  $\mathcal{DRBGC}$ . Then, Proposition 33 and Lemma 34 apply for a measure  $\mu$  with a density  $\phi > 0$  of Class  $\mathcal{C}^1$ . Together, they imply Theorem 20.

**6.5. Two types of renewal equations.** Aaronson considers the invariant measure  $\pi$  under  $\mathcal{S}$  for which  $\pi(\mathcal{J}) = 1$  – exactly the measure associated with the function  $(I - \mathbf{A})^{-1}[\psi]$  of Proposition 29 – and uses the measure  $\pi_{\mathcal{J}}$ ,

$$(73) \quad \pi_{\mathcal{J}}(X) = \pi(X \cap \mathcal{J}).$$

He deals with the measures  $\pi_{\mathcal{J}}(X)$  particularly when  $X = T^{-k}(\mathcal{J})$  and exhibits the following renewal equation (see [A86] page 1042) that relates the two following generating functions :

**Lemma 35.** [Aaronson] *The two generating functions –whose coefficients are defined in terms of the sequence  $\pi_{\mathcal{J}}(T^{-k}(\mathcal{J}))$  that involves the measure  $\pi_{\mathcal{J}}$  defined in (73)– satisfy the renewal equation*

$$(74) \quad \left( \sum_{k \geq 0} \pi_{\mathcal{J}}[T^{-k} \mathcal{J}] v^k \right) \cdot \left( 1 - \sum_{k \geq 0} \pi_{\mathcal{J}} \left[ T^{-k} \mathcal{J} - \left( \bigcup_{\ell < k} T^{-\ell} \mathcal{J} \right) \right] v^k \right) \sim_{v \rightarrow 1} 1.$$

*Proof.* Aaronson first proves that, as soon as the block source  $\mathcal{B}_{\mathcal{J}}$  is  $\psi$ -mixing, the second interval  $\mathcal{J} = [c, 1]$  of the system (where the symbol one is emitted) is a Darling-Kac set<sup>10</sup>. As  $\mathcal{J}$  is a Darling-Kac set, Aaronson obtains the following renewal equation between two functions (here  $\tau$  is a real positive and  $W$  is the waiting time of  $J$ ),

$$(75) \quad \left[ \sum_{k \geq 0} \pi[\mathcal{J} \cap T^{-k}(\mathcal{J})] e^{-\tau k} \right] \left[ \int_{\mathcal{J}} (1 - e^{-\tau W}) d\pi(x) \right] \sim_{\tau \rightarrow 0} \pi(\mathcal{J})^2.$$

The estimate of the integral is

$$\begin{aligned} \int_{\mathcal{J}} (1 - e^{-\tau W}) d\pi(x) &= \sum_{k \geq 0} \int_{\mathcal{J}} \mathbb{1}[W = k] (1 - e^{-\tau W}) d\pi(x) = \sum_{k \geq 0} (1 - e^{-\tau k}) \pi(\mathcal{J} \cap \mathbb{1}[W = k]) \\ &= \pi(\mathcal{J}) - \sum_{k \geq 0} \pi \left( \mathcal{J} \cap \left[ T^{-k} \mathcal{J} - \left( \bigcup_{\ell < k} T^{-\ell} \mathcal{J} \right) \right] \right) e^{-\tau k}. \end{aligned}$$

And finally the left member in Aaronson's renewal equation (75) is (with  $v = e^{-\tau}$ ),

$$\left( \sum_{k \geq 0} \pi[\mathcal{J} \cap T^{-k} \mathcal{J}] v^k \right) \cdot \left( \pi(\mathcal{J}) - \sum_{k \geq 0} \pi \left[ \mathcal{J} \cap \left[ T^{-k} \mathcal{J} - \left( \bigcup_{\ell < k} T^{-\ell} \mathcal{J} \right) \right] \right] v^k \right);$$

In other words, dealing with the measure  $\pi_{\mathcal{J}}$  defined in (73), Aaronson obtains the renewal equation described in (74).  $\square$

We have also obtained, for any measure  $\mu$  with a strictly positive density  $\phi$  of Class  $\mathcal{C}^1$ , a renewal equation between "our" generating functions

$$(1 - v)Q_{\mu}(v), \quad (1 - v)N_{\mu}(v)$$

whose coefficients are defined via the measure  $\mu$ .

There is thus a strong parallelism between the two renewal equations, the previous equation (74) and our renewal equation described in (28).

**Lemma 36.** *The two renewal equations, the one obtained by Aaronson in (74) and ours described in (28) for  $\mu = \nu$  may be deduced from each one to another one.*

*Proof.* Due to Proposition 29(iv).  $\square$

We remark that the renewal equation obtained in the present paper is more general than Aaronson's one since it deals with any measure  $\mu$  with a strictly positive density  $\phi$  of Class  $\mathcal{C}^1$ , and not only with the invariant measure  $\nu$ .

## 7. PROOF OF THE THIRD MAIN RESULT.

We first list in Section 7.1 the main properties of slowly varying functions. Some of them will be used in the next Section 7.2, which states an auxiliary result of interest in the proof of the third main result, which will be provided in Section 7.3.

<sup>10</sup>We do not give a precise definition of the notion of Darling-Kac and  $\psi$ -mixing. For precise definitions, see the papers of Aaronson, and [DK57]

**7.1. Main properties of slowly varying functions.** We collect here definitions and main properties that deal with slowly varying functions. See [BS71], [BGT89] and [GS73] for more detailed studies.

(P1) [Basic definitions] A positive function  $V : ]0, +\infty[$  belongs to  $\mathcal{SV}$  (slowly varying in  $+\infty$ ) iff, for every  $t > 0$ , the ratio  $V(tx)/V(x)$  tends to 1 as  $x \rightarrow \infty$ .

A function  $V$  belongs to  $\mathcal{SV}_0$  (slowly varying in 0) if  $x \mapsto V(1/x)$  belongs to  $\mathcal{SV}$

(P2) A function  $V$  belongs to  $\mathcal{SV}$  iff, for every  $t > 0$ , the convergence in (P1) is uniform wrt to  $t \in [a, b]$  with  $0 < a < b < \infty$

(P3) A function  $V$  belongs to  $\mathcal{SV}$  iff it admits the decomposition

$$V(x) = \zeta(x) \exp \left[ \int_1^x \frac{\epsilon(t)}{t} dt \right],$$

with  $\epsilon(x) \rightarrow 0$ ,  $\zeta(x) \rightarrow C$ , ( $0 < C < \infty$ ) ( $x \rightarrow \infty$ ).

(P4) For  $V \in \mathcal{SV}_0 \cap \mathcal{C}^1$ , the decomposition of (P3) holds with  $\zeta \in \mathcal{C}^1$ ,  $\epsilon \in \mathcal{C}^0$ .

(P5) For any  $V_1 \in \mathcal{SV}_0 \cap \mathcal{C}^1(]0, 1])$ , there exists  $V \in \mathcal{C}^1(]0, 1])$  such that,

$$V(x) \sim V_1(x), \quad \frac{xV'(x)}{V(x)} \rightarrow 0, \quad (x \rightarrow 0).$$

(P6) If  $V \in \mathcal{SV}_0$ , then, for any  $\epsilon > 0$ , there exists  $A_\epsilon$  such that, for  $0 < x < A_\epsilon$ ,

$$x^\epsilon \leq V(x) \leq x^{-\epsilon}.$$

**7.2. An auxiliary result.** The following result plays a central role in the proof of Theorem 23:

**Proposition 37.** *When a sequence  $V_n$  satisfies (29), the sequence  $\tilde{V}_n$  defined as*

$$(76) \quad \tilde{V}_n = \frac{1}{n^{1-\beta}} \sum_{k < n} \frac{V_k}{k^\beta},$$

satisfies  $\tilde{V}_n \sim \frac{1}{1-\beta} V_n$  ( $0 \leq \beta < 1$ ),  $\tilde{V}_n = \Theta(\log n)^{\delta+1}$  ( $\beta = 1$ ).

For any  $\beta \in [0, 1]$ , the sequence  $\tilde{V}_n$  belongs to  $\mathcal{SV}$ .

The estimate of Proposition 37 holds when the sum in Eqn (76) is replaced by the integral: in the book [F71], Theorem 1, p. 281, the precise following result is indeed proven,

When a sequence  $V_n$  satisfies (29) with  $\beta \in [0, 1]$ , the sequence  $\hat{V}_n$  defined as

$$(77) \quad \hat{V}_n = \frac{1}{n^{1-\beta}} \int_1^n \frac{V(x)}{x^\beta} dx,$$

satisfies  $\hat{V}_n \sim \frac{1}{1-\beta} V_n$  ( $0 \leq \beta < 1$ ),  $\hat{V}_n = \Theta(\log n)^{\delta+1}$  ( $\beta = 1$ ).

*Proof.* It is then sufficient to prove the estimate  $\hat{V}_n \sim \tilde{V}_n$ . With Property (P6), and for  $\beta \in [0, 1]$ , the series of general term  $V(k)/k^\beta$  is divergent. With the decomposition

$$\int_1^n \frac{V(x)}{x^\beta} dx = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{V(x)}{x^\beta} dx,$$

it is sufficient to prove that

$$(78) \quad \int_k^{k+1} \frac{V(x)}{x^\beta} dx \sim \frac{V(k)}{k^\beta} \quad (k \rightarrow \infty).$$

With the change of variable  $x = \lambda k$  in the integral over  $[k, k+1]$ ,

$$\int_k^{k+1} \frac{V(x)}{x^\beta} dx = \frac{k}{k^\beta} \int_1^{1+\frac{1}{k}} \frac{V(\lambda k)}{\lambda^\beta} d\lambda,$$

the ratio between the two terms of Eqn (78) equals

$$(79) \quad k \int_1^{1+\frac{1}{k}} \frac{V(\lambda k)}{V(k)} \frac{1}{\lambda^\beta} d\lambda.$$

Now, as  $V \in \mathcal{SV}$ , with Property (P2), the ratio  $V(\lambda k)/V(k)$  tends to 1 as  $k \rightarrow \infty$  uniformly in  $\lambda \in [1, 2]$  and this implies

$$\forall \epsilon > 0, \exists N_\epsilon, \forall k > N_\epsilon, \forall \lambda \in [1, 2], \quad \frac{1-\epsilon}{\lambda^\beta} < \frac{V(\lambda k)}{V(k)} \frac{1}{\lambda^\beta} < \frac{1+\epsilon}{\lambda^\beta}.$$

By integrating with respect to  $\lambda \in [1, 1 + (1/k)]$ , it follows that the ratio of Eqn (79) satisfies for  $\epsilon > 0$  and  $k > N_\epsilon$ ,

$$\frac{k(1-\epsilon)}{1-\beta} \left[ \left(1 + \frac{1}{k}\right)^{1-\beta} - 1 \right] < k \int_1^{1+\frac{1}{k}} \frac{V(\lambda k)}{V(k)} \frac{1}{\lambda^\beta} d\lambda < \frac{k(1+\epsilon)}{1-\beta} \left[ \left(1 + \frac{1}{k}\right)^{1-\beta} - 1 \right].$$

When  $k$  goes to infinity, the ratio of Eqn (79) is between  $(1-\epsilon)[1 + O(1/k)]$  and  $(1+\epsilon)[1 + O(1/k)]$ , and when  $\epsilon$  goes to zero, this concludes the proof.  $\square$

### 7.3. Proof of the third main result Theorem 23.

(A) When  $q_\nu(n) \geq 0$  is of the form (29), the partial sum  $Q_n^{(\nu)}$  of the first  $n$  coefficients of the gf  $Q_\nu(v)$  is written in terms of  $\tilde{V}_n^{(\nu)}$  defined in (76)

$$Q_n^{(\nu)} = \sum_{k < n} q_\nu(n) = n^{1-\beta} \tilde{V}_n^{(\nu)},$$

and Proposition 37 shows that  $\tilde{V}_n^{(\nu)}$  belongs to  $\mathcal{SV}$  for any  $\beta \in [0, 1]$ .

Moreover, with Thm 22, and for any  $\beta \in [0, 1]$ , the function  $Q_\nu(v)$  satisfies (2) of Thm 22 with exponent  $1-\beta \geq 0$  and sequence  $\underline{V}_n^{(\nu)}$ , equal to

$$(80) \quad \underline{V}_n^{(\nu)} = \Gamma(2-\beta) \cdot \tilde{V}_n^{(\nu)}.$$

We consider the function  $\underline{V}^{(\nu)}$  defined from the sequence  $\underline{V}_n^{(\nu)}$  as in Theorem 22. Now, the function  $\hat{Q}_\nu(v) = (1-v)Q_\nu(v)$  satisfies (1) of Thm 22 with the exponent  $-\beta \in [-1, 0]$  and the function  $\underline{V}^{(\nu)}$ .

We then use Thm 20 that relates the function  $\hat{N}_\mu(v) = (1-v)N_\mu(v)$  associated with the generating function  $N_\mu(v)$  of the number of ones and the function  $\hat{Q}_\nu(v)$  via the estimate as the real  $v \rightarrow 1^-$ ,

$$\hat{N}_\mu(v) \sim 1/\hat{Q}_\nu(v).$$

Then  $\hat{N}_\mu(v)$  satisfies (1) of Thm 22 with an exponent  $\beta \in [0, 1]$  and the function  $1/\underline{V}^{(\nu)}$ . This function belongs to  $\mathcal{SV}$  due to Proposition 37 and Eqn (80).

As  $\hat{N}_\mu(v)$  has positive coefficients, whose partial sums of order  $n$  coincide with  $\underline{n}_\mu(n)$ , this entails, with Thm 22, that the sequence  $\underline{n}_\mu(n)$  satisfies (2) of Thm 22 with exponent  $\beta$  and the sequence

$$(1/\Gamma(\beta+1))1/\underline{V}_n^{(\nu)}.$$

Then, using the expression of  $\underline{V}_n^{(\nu)}$  in terms of  $\tilde{V}_n^{(\nu)}$  in (80), the estimate holds,

$$(81) \quad \underline{n}_\mu(n) \sim \frac{n^\beta}{\Gamma(\beta+1)} \frac{1}{\underline{V}_n^{(\nu)}} \sim \frac{n^\beta}{\Gamma(\beta+1)\Gamma(2-\beta)} \frac{1}{\tilde{V}_n^{(\nu)}}$$

Using the expression  $\tilde{V}_n^{(\nu)}$  in terms of  $Q^{(\nu)}(n)$  in (77) leads to the estimates

$$(82) \quad \underline{n}_\mu(n) \sim \frac{1}{\Gamma(\beta+1)\Gamma(2-\beta)} \frac{1}{Q_n^{(\nu)}} n;$$

As  $Q_n^{(\nu)} \rightarrow \infty$  by hypothesis, this proves the first result of Assertion (A).

In the same vein, we apply the Tauberian step (Implication (1)  $\implies$  (2)) to the function  $(1-v)M_\mu(v)$ . As the two generating functions  $N_\mu$  and  $M_\mu$  satisfy, as an indirect consequence of Theorem 20,

$$(1-v)M_\mu(v) \sim \mathcal{E}_\nu(\mathcal{B}) (1-v)N_\nu(v),$$

this gives rise to the estimate of Shannon weights given in Eqn (32)

(B) In the case when  $q_\nu(n) \geq 0$  is of the form (30), the following holds for  $\tilde{V}_n^{(\nu)}$

$$\tilde{V}_n^{(\nu)} \sim \frac{K_\nu}{1-\beta} (\log n)^\delta, \quad (0 \leq \beta < 1), \quad \tilde{V}_n^{(\nu)} = \frac{K_\nu}{\delta+1} (\log n)^{\delta+1}$$

Using then the general estimate (81) for  $\underline{n}_\mu(n)$  leads to the estimates (33).

(C) The partial sum  $Q_n^{(\nu)}$  satisfies,

$$Q_n^{(\nu)} \sim Q_\nu(1), \quad n \rightarrow \infty.$$

Moreover, the function  $Q_\nu(v)$  satisfies (2) of Thm 22 with exponent 0 and the constant function  $Q_\nu(1)$ . The function  $\hat{Q}_\nu(v) = (1-v)Q_\mu(v)$  satisfies (1) of Thm 22 with the exponent  $-1$  and the constant function  $Q_\nu(1)$ .

We will use Theorem 20 that relates the function  $\hat{N}_\mu(v) = (1-v)N_\mu(v)$  associated with the generating function  $N_\mu(v)$  of the number of ones and the function  $\hat{Q}_\nu(v)$  via the estimate as the real  $v \rightarrow 1^-$ ,

$$\hat{N}_\mu(v) \sim 1/\hat{Q}_\nu(v).$$

Then  $\hat{N}_\mu(v)$  satisfies (1) of Thm 22 with the exponent 1 and the function  $1/\hat{Q}_\nu(1)$ . As  $\hat{N}_\mu(v)$  has positive coefficients, whose partial sums of order  $n$  coincide with  $\underline{n}_\mu(n)$ , this entails, with Thm 22, the estimate

$$\underline{n}_\mu(n) \sim \frac{1}{Q_\nu(1)} n, \quad \underline{n}_\mu(n) \sim \frac{1}{\mathbb{E}_\nu[W]} n.$$

This ends the proof of our main third result, Theorem 23.

## 8. THE CLASS *DRIS*

This Section aims at exhibiting explicit instances of sources of the Class *DRBGC* that will be found in a subclass of the class *DRL*, denoted as *DRIS*. The short name *DRIS* stands for dynamical recurrent sources which exhibits an indifferent fixed point with slowly varying behaviour. Such a class indeed gathers systems for which 0 is an indifferent fixed point for branch  $a$ , with a precise description of the contact at  $x = 0$  between the graph of map  $a$  with its tangent (the line  $a(x) = x$ ) which involves slowly varying functions.

Section 8.1, with Definition 39, precisely defines the (strong) notion of indifferent fixed point and the associated subclass *DRIS*. Section 8.2 compares our definition with other definitions of the same spirit that are found in the literature. Then, Section 8.3 uses a result due to Aaronson (for which we provide a precise proof) and obtains first results on the wtd sequence  $q(n)$ , that provide a first estimate on the convergence abscissa of the Dirichlet series which intervenes in Item (a) of the Good Class (see Definition 17). Finally, Section 8.4 with its Theorem 42, proves the main result of the Section: the block source of a *DRIS* source belongs to the Good Class.

**8.1. Notion of strong indifferent fixed point : the class *DRIS*.** Consider a source of the Class *DRL*, for which the branch  $a$  admits 0 as an indifferent fixed point. In this case, the equalities  $a(0) = 0$  and  $a'(0) = 1$  hold, and we introduce a more precise notion, that deals with the set  $\mathcal{SV}_0$  of slowly varying functions at  $x = 0$ , in order to describe the behaviour of the derivative  $a'$  near 0.

We consider a particular subclass of  $\mathcal{SV}_0$ , already mentioned in (P5) of Section 7.1, and introduced in [dB59], whose definition is recalled here.

**Definition 38.** *The class  $\mathcal{SV}_0^*$  gathers the functions  $V \in \mathcal{SV}_0$  of class  $\mathcal{C}^1([0, 1])$  for which*

$$\epsilon(x) = \frac{xV'(x)}{V(x)} \rightarrow 0 \quad (x \rightarrow 0).$$

A result due to de Bruijn, and proven in [BGT89] page 15, states the following:

for any  $V_1 \in \mathcal{SV}_0 \cap \mathcal{C}^1([0, 1])$ , there exists  $V \in \mathcal{SV}_0^*$  that satisfies  $V(x) \sim V_1(x)$  as  $x \rightarrow 0$ .

Most of the natural functions of  $\mathcal{SV}_0 \cap \mathcal{C}^1([0, 1])$  belong to  $\mathcal{SV}_0^*$ , for instance the functions of  $\mathcal{C}^1([0, 1])$  or any power (positive or negative)  $x \mapsto (\log x)^\delta$ .

The following definition deals with the set  $\mathcal{SV}_0^*$  and provides the notion of order of an indifferent fixed point.

**Definition 39** (indifferent fixed point of order  $\gamma > 0$ ). *A map  $a$  defined on the unit interval  $\mathcal{I} = [0, 1]$ , and of class  $\mathcal{C}^2([0, 1])$  satisfying  $a(0) = 0$  has an indifferent fixed point at  $x = 0$  of order  $\gamma > 0$  if there exists a positive function  $V \in \mathcal{SV}_0^*$  for which  $a$  is written, for any  $x \in \mathcal{I}$ , as*

$$(83) \quad a(x) = x - u(x), \quad u'(x) = x^\gamma V(x), \quad u'(x) \in [0, 1],$$

with the conditions at  $x = 0$ ,  $u(0) = 0$ ,  $u'(0) = 0$ . One lets  $v(x) = u(x)/x$  so that

$$a(x) = x(1 - v(x)).$$

We now describe first properties of such a branch  $a$ .

**Lemma 40.** *When  $a$  has an indifferent fixed point of order  $\gamma > 0$ , with  $a'(x) = 1 - x^\gamma V(x)$  and  $V \in \mathcal{SV}_0^*$ , the following holds:*

- (i)  $a(x) \in [0, 1]$  for  $x \in [0, 1]$ .
- (ii) there exists a neighborhood of 0, say  $[0, x_0]$  on which the fonction  $a$  is concave
- (iii) the function  $v$  satisfies

$$v(x) = \frac{x^\gamma}{\gamma + 1} V_1(x), \quad V_1(x) \sim V(x) \quad (x \rightarrow 0)$$

and  $V_1$  is slowly varying

*Proof.* (i) First, since  $u'(x) \in [0, 1]$ , it is the same for  $1 - u'(x)$  and

$$a(x) = x - u(x) = \int_0^x (1 - u'(t)) dt \in [0, 1] \text{ for } x \in [0, 1].$$

(ii) The derivative  $u''(x)$  satisfies

$$u''(x) = \gamma x^{\gamma-1} V(x) + x^\gamma V'(x) = \gamma x^{\gamma-1} V(x) [1 + \epsilon(x)], \quad \epsilon(x) = \frac{1}{\gamma} \frac{xV'(x)}{V(x)}$$

As  $V \in \mathcal{SV}_0^*$ , this entails

$$a''(x) \sim -\gamma x^{\gamma-1} V(x) \quad (x \rightarrow 0).$$

(iii) with integration by parts, the function  $u$  satisfies

$$u(x) = \int_0^x t^\gamma V(t) dt = \left[ \frac{t^{\gamma+1}}{\gamma+1} V(t) \right]_0^x - \int_0^x \frac{t^{\gamma+1}}{\gamma+1} V'(t) dt.$$

This entails

$$\left| u(x) - \frac{x^{\gamma+1}}{\gamma+1} V(x) \right| \leq \frac{\epsilon_1(x)}{\gamma+1} \int_0^x t^\gamma V(t) dt = \frac{\epsilon_1(x)}{\gamma+1} u(x),$$

where  $\epsilon_1(x) = \max_{t \in [0, x]} \epsilon(t)$ ,  $\epsilon(t) = \frac{xV'(t)}{V(t)}$  tends to 0 as  $x \rightarrow 0$

and thus Item (c) holds.  $\square$

Remark: The functions  $v$  and  $(\gamma + 1)u'$  are thus equivalent as  $x \rightarrow 0$ .

**Definition 41.** We denote by  $\mathcal{DRIS}(\gamma)$  the subset of the class  $\mathcal{DRI}$  which gathers the systems for which the first inverse branch  $a$  admits an indifferent fixed point of order  $\gamma > 0$ , and we let

$$\mathcal{DRIS} = \bigcup_{\gamma > 0} \mathcal{DRIS}(\gamma).$$

As explained in Section 3.3, when the branch  $a$  admits an indifferent point of order  $\gamma > 0$ , there are two main cases for the derivative  $b'(0)$  of branch  $b$ : – a strong case, called the case ( $s$ ), where  $|b'(0)| < 1$ , – a weak case, called the case ( $w$ ) where  $b'(0) = -1$ . According to the two possible cases for the branch  $b$ , there are thus two subclasses

$$\mathcal{DRIS}_s(\gamma) \text{ and } \mathcal{DRIS}_w(\gamma) .$$

We have seen that the Farey system has a branch  $b$  of the case ( $w$ ).

The importance of the  $\mathcal{DRIS}$  Class in this paper is due to the following Theorem 42, which will be proven in Section 8.4.

**Theorem 42.** *When the source  $\mathcal{S}$  belongs to the  $\mathcal{DRIS}$  Class, its block source  $\mathcal{B}$  belongs to the Good Class, and the inclusion  $\mathcal{DRIS} \subset \mathcal{DRBGC}$  holds.*

**8.2. About our Class  $\mathcal{DRIS}$ .** The idea of dealing with a direct branch  $A$  of class  $\mathcal{C}^2$  with an indifferent point at 0 seems first due to Manneville and Pommeau [PM80]. Then, many authors deal with this type of sources, and make precise the notion of indifferent fixed point they use. Thaler in [T83] gives a precise definition, and obtains precise results on these sources that will be used later on by Aaronson [A86]. Then, Holland [H04] deals with maps  $T$  whose direct branches  $A$  are of class  $\mathcal{C}^2$  and written as

$$A(x) = x(1 + x^\gamma \underline{V}(x)) \text{ with } \underline{V} \text{ of slow varying.}$$

We remark that all the tools Holland needs in the study of “his” class can be in fact expressed with the inverse branch  $a$ .

This is why we directly define our class  $\mathcal{DRIS}$  via the inverse branch  $a$ , and more precisely via the derivative  $a'$  of the inverse branch  $a$ . Then, there are two expressions of the branch  $A$  or its inverse  $a$ ,

$$\begin{aligned} a'(x) &= 1 - x^\gamma V(x) \quad [\text{the present approach}], \\ A(x) &= x(1 + x^\gamma \underline{V}(x)), \quad [\text{Holland's approach}]. \end{aligned}$$

Via Implicit Function Theorem, there thus exists a relation between  $V$  and  $\underline{V}$  that is however not straightforward: this explains that the computations made by Holland are often less direct than ours, notably when he deals with convexity.

**8.3. First results on the waiting time in the class  $\mathcal{DRIS}$ .** Relation (83) gives rise to a recurrence for the sequence  $q_\tau(n) = \tau[W > n] = a^n(1)$  associated with the distribution of the waiting time with respect to the uniform measure  $\tau$ ,

$$(84) \quad q_\tau(0) = 1 \quad q_\tau(n+1) = q_\tau(n)(1 - v(q_\tau(n))).$$

In the sequel of this section, the index  $\tau$  is omitted.

The next proposition, based on a result due to Aaronson, provides first estimates on the behaviour of the wtd sequences  $q(n)$  and  $r(n)$ . This will be useful for describing the behaviour of the Dirichlet series  $\Delta(s)$  which plays an important role in Item (a) of block sources.

**Theorem 43.** *For a source of the  $\mathcal{DRIS}(\gamma)$  Class, with  $\gamma > 0$ , the following holds:*

(i) *The distribution of the waiting time  $W$  satisfies,*

$$v(q(n)) \sim \frac{1}{\gamma n} \quad (n \rightarrow \infty).$$

where  $v$  is the function involved in Definition 39

- (ii) For every  $\varepsilon \in (0, \gamma)$ , there exists  $N = N_\varepsilon$  and constants  $A, \tilde{A} > 0$  for which, for any  $n \geq N$ ,

$$\tilde{A} \left(\frac{1}{n}\right)^{1/(\gamma-\varepsilon)} \leq q(n) \leq A \left(\frac{1}{n}\right)^{1/(\gamma+\varepsilon)} .$$

In particular, the mean  $\mathbb{E}[W]$  is infinite for  $\gamma > 1$  and is finite for  $\gamma < 1$ .

- (iii) When  $\gamma < 1$ , for any  $\hat{\gamma}$  with  $\gamma < \hat{\gamma} < 1$ , there exist an integer  $L(\hat{\gamma})$  and a constant  $A(\hat{\gamma})$  such that for any  $\ell > L(\hat{\gamma})$ , the inequality holds

$$(85) \quad \ell \leq A(\hat{\gamma}) q(\ell)^{-\hat{\gamma}} .$$

- (iv) There exists  $N$  for which, for any  $n \geq N$ , the estimates hold

$$r(n) = q(n+1) - q(n) \leq \left(\frac{1}{n}\right)^{1+(2/(2\gamma+1))} \quad v(q(n)) - v(q(n+1)) \sim \frac{1}{\gamma} \frac{1}{n^2} .$$

- (v) The convergence abscissa of the Dirichlet series  $\Delta_\tau(s) = \sum_{m \geq 1} r(m)^s$  is less than  $[1 + (2/(2\gamma+1))]^{-1}$  and is strictly less than 1.

*Proof.* (i) For a regularly varying map  $v(x) = x^\gamma V_1(x)$  with index  $\gamma$  at  $x = 0$ , we use (84) and the two estimates

$$\frac{1}{v(q(n+1))} = \frac{1}{q(n+1)^\gamma V_1(q(n+1))} \quad \frac{1}{v(q(n))} = \frac{1}{q(n)^\gamma V_1(q(n))} .$$

This entails

$$\frac{1}{v(q(n+1))} = \frac{1}{v(q(n))} \frac{1}{(1-v(q(n)))^\gamma} \cdot \frac{V_1(q(n))}{V_1(q(n+1))} .$$

Then, the difference satisfies,

$$(86) \quad \frac{1}{v(q(n+1))} - \frac{1}{v(q(n))} = \frac{1}{v(q(n))} \left[ \frac{1}{(1-v(q(n)))^\gamma} \cdot \frac{V_1(q(n))}{V_1(q(n+1))} - 1 \right] .$$

For a general slowly varying map  $V_1$ , the asymptotic study of the ratio

$$V_1(q(n))/V_1(q(n+1))$$

is thus central and Aaronson provides the following estimate, for which we recall the proof:

**Lemma 44.** Consider the sequence  $q(n)$  defined as in (84), with  $v(x) = x^\gamma V_1(x)$  and  $V_1 \in \mathcal{SV}_0$  of class  $\mathcal{C}^1$ . Then

$$(87) \quad \frac{V_1(q(n))}{V_1(q(n+1))} = 1 + o(v(q(n))) .$$

*Proof.* When  $V_1$  is of class  $\mathcal{C}^1$ , with  $V_1(0) \neq 0$ , one uses Property (P3) and (P4) in 7.1 to get

$$V_1(x) = \zeta(x) \exp \left[ \int_x^1 \frac{\epsilon(t)}{t} dt \right], \quad \text{with } \zeta \in \mathcal{C}^1, \quad \epsilon \in \mathcal{C}^0, \quad \lim_{t \rightarrow 0} \epsilon(t) = 0 .$$

Then, the estimates hold

$$(88) \quad \log \frac{V_1(q(n))}{V_1(q(n+1))} = \log \frac{\zeta(q(n))}{\zeta(q(n+1))} + o \left( \log \frac{q(n)}{q(n+1)} \right) .$$

The estimate

$$\log \frac{q(n)}{q(n+1)} = -\log(1-v(q(n))) \sim v(q(n)) ,$$

entails that the second term in (88) is  $o(v(q(n)))$ . Moreover,

$$\text{as } \zeta \in \mathcal{C}^1, \text{ with } \zeta(x) = C + O(|x|), \quad C \neq 0,$$

the first term in (88) satisfies,

$$\log \frac{\zeta(q(n))}{\zeta(q(n+1))} = O(\zeta(q(n+1)) - \zeta(q(n))) = O(q(n+1) - q(n)) = O(q(n)v(q(n)))$$

and it is of smaller order than the second term. This entails an estimate  $o(v(q(n)))$  for (88) and thus an estimate for (87) as in Lemma 44.  $\square$

Using this last estimate in (86) leads to

$$\begin{aligned} \frac{1}{v(q(n+1))} - \frac{1}{v(q(n))} &= \frac{1}{v(q(n))} \left[ \frac{1}{(1-v(q(n)))^\gamma} \cdot \frac{V_1(q(n))}{V_1(q(n+1))} - 1 \right] \\ &= \frac{1}{v(q(n))} \left( [1 + \gamma v(q(n) + o(v(q(n)^2))] [1 + o(v(q(n)))] - 1 \right) = \gamma + o(1). \end{aligned}$$

We take the sum of the previous estimates, and obtain

$$\frac{1}{v(q(n))} = \gamma n + o(n), \quad (n \rightarrow \infty), \quad v(q(n)) \sim \frac{1}{\gamma n} \quad \gamma_2 \frac{1}{n} \leq v(q(n)) \leq \gamma_1 \frac{1}{n},$$

with  $\gamma_1$  and  $\gamma_2$  arbitrary close to  $1/\gamma$ .

(ii) The previous estimate given in (i) leads to

$$(89) \quad q(n)^\gamma V_1(q(n)) \sim \frac{1}{\gamma n} \quad q(n) \sim \left( \frac{1}{\gamma n} \right)^{1/\gamma} \left( \frac{1}{V_1(q(n))} \right)^{1/\gamma}.$$

As  $V_1$  belongs to  $\mathcal{SV}_0$ , it is the same for  $1/V_1$ , and  $1/V_1(q(n))$  satisfies, for any  $\epsilon > 0$ , for  $N$  larger than  $N_\epsilon$ ,

$$q(n)^\epsilon \leq \frac{1}{V_1(q(n))} \leq q(n)^{-\epsilon}.$$

This entails, with (89), the two inequalities, as soon as  $N$  is larger than  $N_\epsilon$ ,

$$\tilde{A}_\epsilon \left( \frac{1}{n} \right)^{1/(\gamma-\epsilon)} \leq q(n) \leq A_\epsilon \left( \frac{1}{n} \right)^{1/(\gamma+\epsilon)}.$$

When  $\gamma > 1$ , we choose  $\epsilon$  such that  $\epsilon < \gamma - 1$ . This entails  $1/(\gamma - \epsilon) < 1$  and the divergence of  $\mathbb{E}[W]$  follows. When  $\gamma < 1$ , we choose  $\epsilon$  such that  $\gamma < \hat{\gamma} = \gamma + \epsilon < 1$ . The convergence of  $\mathbb{E}[W]$  follows.

(iii). Clear

(iv) We choose  $\epsilon = 1/2$ . This leads, with Items (i) and (ii), for  $n \geq N$ , and for some constant  $A$ , to the estimate

$$q(n) - q(n+1) = q(n) \left( 1 - (1 - v(q(n))) \right) \leq A \left( \frac{1}{n} \right)^{1+2/(2\gamma+1)}.$$

In the same vein, for some constant  $B$

$$\begin{aligned} v(q(n)) - v(q(n+1)) &= v(q(n)) \left( 1 - \frac{q(n+1)^\gamma V_1(q(n+1))}{q(n)^\gamma V_1(q(n))} \right) \\ &= v(q(n)) \left( 1 - (1 - v(q(n)))^\gamma \frac{V_1(q(n+1))}{V_1(q(n))} \right). \end{aligned}$$

Using Lemma 44 and the same principles as in the proof of Assertion (i), one obtains

$$v(q(n)) - v(q(n+1)) \sim \gamma v(q(n))^2 \sim (1/\gamma)(1/n^2).$$

(v) Clear.  $\square$

8.4. **Proof of the inclusion  $\mathcal{DRIS} \subset \mathcal{DRBGC}$ .** This Section is devoted to the proof of Theorem 42.

*Proof.* Property (a) of the Good Class holds from Item (iv) of Theorem 43.

*Property (b) of the Good Class: strict contraction.* This property holds for the whole Class  $\mathcal{DR}$ – in a strong form– and thus, in particular, for the class  $\mathcal{DRI}$ .

**Lemma 45.** *For a source of the Class  $\mathcal{DRI}$ , the sequence  $\eta_n$  of Definition 17 (b) satisfies the following<sup>11</sup>:*

(a) *For a source of the Class  $\mathcal{DRIS}$ , one has  $\eta_1 < 1$ , whereas for a source of the Class  $\mathcal{DRI}_w$ , one has  $\eta_1 = 1$  and  $\eta_2 < 1$ .*

(b) *For any source in the Class  $\mathcal{DRI}$ , the inequalities hold for  $n \geq 2$ ,*

$$\eta_n \leq \eta_2^{1+\lfloor (n-2)/2 \rfloor} \quad \text{with} \quad \eta_2 = \min(\eta_1^2, \eta_2) < 1 \quad .$$

*Proof.*

Item (a). The derivative  $g'_m$  of the branch  $g_m = a^{m-1} \circ b$  of the source  $\mathcal{B}$  satisfies

$$(90) \quad |g'_m(x)| = |b'(x)| \quad (m = 1), \quad |g'_m(x)| = |b'(x)| \prod_{\ell=1}^{m-1} |a'(a^{\ell-1} \circ b(x))| \quad (m \geq 2).$$

With the intervals  $\mathcal{J}_\ell$  defined in Proposition 13(i) for  $\ell \geq 1$ , and  $\mathcal{J}_0 = \mathcal{I}$ , we define

$$\alpha_m = \max\{a'(x) \mid x \in \mathcal{J}_m\}, \quad (m \geq 1), \quad \beta_m = \max\{|b'(x)| \mid x \in \mathcal{J}_m\}, \quad (m \geq 0).$$

For  $m \geq 1$ , the inequalities  $\alpha_m < 1$  and  $\beta_m < 1$  lead to the bounds

$$\gamma_m = \prod_{\ell=1}^{m-1} \alpha_\ell < \alpha_1, \quad \sup_{m \geq 1} \beta_m \leq 1.$$

In the case of a system in  $\mathcal{DRIS}$ , the first factor in (90) is at most  $\beta_0 < 1$  whereas the second factor is at most equal to  $\gamma_m$  and thus at most  $\alpha_1 \leq 1$ .

In the case of a system in  $\mathcal{DRI}_w$ , we consider the branches of depth 2, of the form  $g_m \circ g_n$  associated with an integer pair  $(m, n)$ , let

$$A(m, n) := \sup\{|(g_m \circ g_n)'(x)| \mid x \in [0, 1]\}, \quad m \geq 1, n \geq 1$$

and consider four cases:

$$(m = n = 1), \quad (m \geq 2, n = 1), \quad (n \geq 2, m = 1), \quad (m \geq 2, n \geq 2).$$

Then, there is a bound for  $A(m, n)$  in each of the four cases, as follows:

$$\begin{aligned} (m = 1, n = 1) : & \beta_1 & (m \geq 2, n = 1) : & \alpha_1 \\ (m = 1, n \geq 2) : & \beta_n \alpha_1 & (m \geq 2, n \geq 2) : & \alpha_1^2. \end{aligned}$$

This entails

$$\eta_2 = \sup\{A(m, n) \mid m, n \geq 1\} \leq \sup(\beta_1, \alpha_1, \alpha_1 \sup_{n \geq 2} \beta_n, \alpha_1^2) \leq \max(\beta_1, \alpha_1) < 1.$$

Item (b). Clear with multiplicativity of derivatives. □

*Property (c) of the Good Class : distortion.*

For each branch  $a, b$  we denote by  $\delta(a), \delta(b)$  the finite bounds,

$$(91) \quad \delta(a) = \max_{(x,y) \in \mathcal{I}^2} \frac{|a'(x)|}{|a'(y)|}, \quad \delta(b) = \max_{(x,y) \in \mathcal{I}^2} \frac{|b'(x)|}{|b'(y)|}.$$

We also remark that the distortion ratio is expressed as a derivative

$$(92) \quad \frac{|h''(x)|}{|h'(x)|} = \frac{d}{dx} \left( \log |h'(x)| \right) \quad \text{and thus} \quad \left| \log \frac{|h'(x)|}{|h'(y)|} \right| = \left| \int_x^y \frac{|h''(u)|}{|h'(u)|} du \right|.$$

<sup>11</sup>The Classes  $\mathcal{DRIS}$  and  $\mathcal{DRI}_w$  are defined in Section 3.3.

With (90), the derivative  $g'_m$  of the inverse branch  $g_m = a^{m-1} \circ b$  satisfies,

$$(93) \quad \log \frac{|g'_m(x)|}{|g'_m(y)|} = \log \frac{|b'(x)|}{|b'(y)|} + \sum_{\ell=0}^{m-2} A_\ell, \quad A_\ell = \log \frac{|a'(a^\ell \circ b(x))|}{|a'(a^\ell \circ b(y))|}.$$

Each term  $A_\ell$  involves two points that belong to

$$\mathcal{J}_{\ell+1} = [a^\ell \circ b(1), a^\ell \circ b(0)] = [q(\ell+1), q(\ell)].$$

Consider the point  $x_0$  in Lemma 40 (item (ii)). As each interval  $\mathcal{J}_\ell$  satisfies  $\mathcal{J}_{\ell+1} \subset [0, q(\ell)]$  and since  $q(\ell) \rightarrow 0$ , there is an integer  $M_1$  for which, for any  $\ell \geq M_1$ , one has  $q(\ell) \leq x_0$ . Then, we may apply (92) to any pair of points of the interval  $\mathcal{J}_\ell$ . Moreover  $a'(0) = 1$  and  $a'(x) > 0$  on  $[0, x_0]$ . This entails the estimate for  $\ell \geq M_1$ ,

$$(94) \quad |A_\ell| \leq K \left| \int_{a^\ell \circ b(x)}^{a^\ell \circ b(y)} a''(u) du \right| \leq K |v(q(\ell) - v(q(\ell+1)))|.$$

We apply Item (iii) of Theorem 43, and, choosing  $M = \max(M_1, N)$ , the sum of estimates in (94) for  $\ell \geq M$  is of order  $\Theta[M^{-1}]$ .

For  $\ell \leq M$ , using the bound (91), the sum for indices  $\ell \leq M$  in (93) is less than

$$M \log \delta(a) + \log \delta(b).$$

This entails, for any  $(x, y) \in \mathcal{I}^2$  and for any integer  $m$ , the final distortion bound,

$$\log \frac{|g'_m(x)|}{|g'_m(y)|} \leq (\log \delta(b) + M \log \delta(a)) + \frac{K}{M}$$

that leads to the result. □

## 9. THE CLASS $\mathcal{DRIL}$ .

We introduce in Section 9.1 the subclass  $\mathcal{DRIL}$ , where the letter  $\mathcal{L}$  is relative to a logarithm behaviour of the inverse branch  $a$ . Proposition 48 of Section 9.2 provides precise estimates for the wtd sequence. Then, Section 9.3 answers the main question of the paper and explains how to find a source of the Class  $\mathcal{DRIL}$  with a prescribed Shannon weight and Section 9.4 made important remarks on this result. The Section ends in Section 9.5 with the description of some explicit instances of the subclass  $\mathcal{DRIL}$ , with their graphs.

**9.1. Definition of the Class  $\mathcal{DRIL}$ .** We consider a subclass of the class  $\mathcal{DRI}$  that gathers functions  $u$  in (83) whose derivative  $u'$  is of logarithmic type. This explains the “ $\mathcal{L}$ ” at the end of the name of the subclass.

**Definition 46.** *The class  $\mathcal{DRIL}(\gamma, \delta)$  is the subset of the class  $\mathcal{DRI}$  which gathers the functions  $u \in \mathcal{C}^2$  whose derivative  $u'$  satisfies  $u'(x) \in [0, 1]$  and is written, instead of (83), as  $u'(x) = x^\gamma V_\delta(x)$ , with*

$$(95) \quad V_\delta(x) = A |\log \frac{x}{2}|^\delta, \quad (\delta \neq 0), \quad V_0(x) \in \mathcal{C}^1, \quad V_0(x) > 0 \quad (\delta = 0),$$

with some constant  $A$ , and some function  $V_0$ .

The following inclusions hold,

$$(\gamma = 1, \delta \leq 0) \quad \mathcal{DRIL}(1, \delta) \subset \mathcal{DRIS}(1), \quad (\gamma > 1, \delta \in \mathbb{R}) \quad \mathcal{DRIL}(\gamma, \delta) \subset \mathcal{DRIS}(\gamma),$$

and lead us to the set  $\Gamma_S$  for possible parameters  $(\gamma, \delta)$  of the source  $\mathcal{S}$ , together with the subset  $\mathcal{DRIL}$ ,

$$(96) \quad \Gamma_S = ([1, +\infty[\times\mathbb{R}) \cup (\{1\}\times] - \infty, 0]), \quad \mathcal{DRIL} = \bigcup_{(\gamma, \delta) \in \Gamma_S} \mathcal{DRIL}(\gamma, \delta).$$

**Lemma 47.** *The following holds*

- (i) For any  $(\gamma, \delta) \in \Gamma_S$ , there are choices for the constant  $A$ , or the function  $V_0$ , which entail that

$$u'(x) = x^\gamma V_\delta(x) = Ax^\gamma |\log \frac{x}{2}|^\delta \in [0, 1] \text{ for } x \in \mathcal{I}.$$

For  $\delta \neq 0$ , any  $A < A_{\gamma, \delta}^{-1}$ , with  $A_{\gamma, \delta}$  defined in (98), is a valid choice.

- (ii) For any  $\delta \neq 0$ , the functions  $u$  and  $v$  satisfy

$$\frac{u(x)}{x} = v(x) = \frac{A}{\gamma+1} x^\gamma |\log \frac{x}{2}|^\delta \left[ 1 + O\left(\frac{1}{|\log x|}\right) \right].$$

For (strictly) positive integer values of  $\delta$ , the two functions  $u(x)$ , and  $v(x) = u(x)/x$  are written as the product of the power function  $x^\gamma$  by a polynomial  $P_\delta$  in the variable  $|\log \frac{x}{2}|$  of degree at most  $\delta$ .

*Proof.* First remark that  $u'$  is well defined on  $[0, 1]$  for any  $\delta$ .

- (i) When  $\delta \neq 0$ , we make precise the conditions on the constant  $A$ . The derivatives satisfy,

$$V'_\delta(x) = \delta \left(\frac{-1}{x}\right) (-\log \frac{x}{2})^{\delta-1} = -\frac{\delta}{x} (-\log \frac{x}{2})^{\delta-1}, \quad \frac{xV'_\delta(x)}{V_\delta(x)} = -\delta |\log \frac{x}{2}|^{-1} = -\frac{\delta}{\log \frac{x}{2}},$$

$$(97) \quad \text{and thus } u''(x) = x^{\gamma-1} V_\delta(x) [\gamma + \epsilon(x)], \quad \text{with } \epsilon(x) = \frac{\delta}{\log \frac{x}{2}}.$$

For  $\gamma = 1$ , we ask a finite limit for  $V_\delta$  as  $x \rightarrow 0$ , and thus  $\delta \leq 0$ . For any value of  $\gamma \geq 1$ , there is a zero for  $u''$  at the point  $x_0$  for which

$$\log \frac{x_0}{2} = -\delta/\gamma, \text{ i.e., } x_0 = 2e^{-\delta/\gamma}.$$

When  $x_0 \leq 1$  the function  $u'$  has a maximum at  $x_0$ , whereas, for  $x_0 \geq 1$ , the maximal value of  $u'$  is attained at  $x = 1$ . The condition  $x_0 \leq 1$  is

$$e^{-\delta/\gamma} \leq 1/2 \text{ or, equivalently, } \delta \geq (\log 2) \gamma > 0.$$

In all the cases with  $\delta \neq 0$ , the maximal value of  $u'$  is  $A \times A_{\gamma, \delta}$ , where

$$(98) \quad A_{\gamma, \delta} = 2^\gamma e^{-\delta} (\delta/\gamma)^\delta, \quad \delta \geq (\log 2) \gamma; \quad A_{\gamma, \delta} = (\log 2)^\delta, \quad \text{otherwise.}$$

For  $\delta \neq 0$ , we are thus led to deal with functions  $u_{\gamma, \delta}$  associated with a constant  $A$  which satisfies  $A < A_{\gamma, \delta}^{-1}$  and  $A_{\gamma, \delta}$  defined in (98).

For  $\delta = 0$ , we ask the function  $u'_{\gamma, 0} = x^\gamma V_0(x)$  to belong to the interval  $[0, 1]$ .

- (ii) Due to (97), this is Item (c) of Lemma 40 in the case when  $\epsilon(x) = \delta(\log \frac{x}{2})^{-1}$  and thus  $\epsilon_1(x) = \epsilon(x)$ . The explicit expressions are obtained with successive integrations by parts.  $\square$

**9.2. Precise estimates of the wtd sequence.** For a system of the subclass  $\mathcal{DRIL}$ , the block source  $\mathcal{B}$  belongs to the Good Class. However, there are no precise results on the asymptotic behaviour of its wtd sequence. We only know the asymptotic behaviour of the sequence  $v(q(n))$  from Theorem 43(i). This is why we introduce the subclass  $\mathcal{DRILC}$  where it is possible to obtain precise results on the asymptotic behaviour of the wtd sequence. This is the main interest of the  $\mathcal{DRILC}$  Class.

**Proposition 48.** For a source  $\mathcal{S}$  of the  $\mathcal{DRILC}(\gamma, \delta)$  Subclass, the wtd  $q(n)$  satisfies

$$q(n) \sim \gamma^{(\delta-1)/\gamma} n^{-1/\gamma} (\log n)^{-\delta/\gamma} \quad (\delta \neq 0), \quad q(n) \sim Bn^{-1/\gamma} \quad (\delta = 0).$$

*Proof.* We start with the general estimate for  $v(q(n))$  obtained in Item (i) of Theorem 43. We want to show that we can invert this relation asymptotically, yielding the asymptotics for  $q(n)$ .

For  $\delta = 0$ , the estimate is direct. When  $\delta \neq 0$ , one has

$$v(x) = x^\gamma V_1(x), \quad V_1(x) = A |\log \frac{x}{2}|^\delta \left[ 1 + O\left(\frac{1}{|\log x|}\right) \right].$$

Consider pairs  $(x, y)$  for which  $v(y) \sim x$  with  $x, y \rightarrow 0$ . Such pairs  $(x_n, y_n)$  exist,

$$(99) \quad y_n = q(n), \quad x_n = 1/(\gamma n)$$

Notice that:

$$x \sim v(y) = A y^\gamma |\log \frac{y}{2}|^\delta (1 + O(\frac{1}{|\log y|})), \quad x^{1/\gamma} \sim A^{1/\gamma} y |\log \frac{y}{2}|^{\delta/\gamma}$$

$$\frac{1}{\gamma} \log x = \log y - \frac{\delta}{\gamma} \log |\log \frac{y}{2}| + o(1) = \log y (1 + \epsilon(y))$$

with  $\epsilon(y) \rightarrow 0$  as  $y \rightarrow 0$ , so that the estimates hold, (as  $x, y$  tend to 0)

$$\log \frac{y}{2} \sim \log y \sim \frac{1}{\gamma} \log x, \quad y \left| \frac{1}{\gamma} \log x \right|^{\delta/\gamma} \sim A^{-1/\gamma} x^{1/\gamma},$$

$$\text{and finally } y \sim A^{-1/\gamma} \left( \frac{1}{\gamma} \right)^{-\delta/\gamma} x^{1/\gamma} |\log x|^{-\delta/\gamma}.$$

Applying to the pairs  $(x_n, y_n)$  given in (99) leads to the estimate of Proposition 48 in the case  $\delta \neq 0$ .  $\square$

**9.3. Exhibiting a *DRIL* source with a prescribed Shannon weight.** We associate with a source  $\mathcal{S}$  of the subset *DRIL* the set  $\Gamma_S$  of parameters  $(\gamma, \delta)$  defined in (96) together with the pair  $(\beta_Q, \delta_Q)$  of parameters that describe the wtd  $q(n)$  of such a source  $\mathcal{S}$ . Proposition 48 exhibits an injective map defined on  $\Gamma_S$

$$(S \rightarrow Q) : (\gamma_S, \delta_S) \mapsto (\beta_Q, \delta_Q), \quad \beta_Q := 1/\gamma_S, \quad \delta_Q = -\delta_S/\gamma_S,$$

whose image is

$$\Gamma_Q^* = (]0, 1[ \times \mathbb{R}) \cup (\{1\} \times [0, \infty[).$$

The set  $\Gamma_Q^*$  is (strictly) contained in the set  $\Gamma_Q$  defined in (31), recalled here

$$\Gamma_Q = (]0, 1[ \times \mathbb{R}) \cup (\{1\} \times [-1, \infty[) \cup (\{0\} \times [-\infty, 0]).$$

In other terms, there does not exist a source of the Class *DRIL* whose wtd parameters  $(\beta, \delta)$  satisfy

$$(\beta = 1, \delta \in [-1, 0]) \quad \text{or} \quad \beta = 0.$$

Theorem 23 applies to sources  $\mathcal{S}$  in the Class *DRIL*, and the injective map described in (36)

$$(Q \rightarrow M) : (\beta_Q, \delta_Q) \mapsto (\beta_M, \delta_M)$$

is defined on  $\Gamma_Q^*$ . Finally, for a source  $\mathcal{S}$  of the subclass *DRIL*, there is an injective map  $(S \rightarrow M)$  defined on  $\Gamma_S$  equal to the composition of the two maps  $(S \rightarrow Q)$  and  $(Q \rightarrow M)$ . One has

$$(S \rightarrow M) : (\gamma_S, \delta_S) \mapsto (\beta_M, \delta_M), \quad \text{with}$$

$$\beta_M = 1/\gamma_S, \quad \delta_M = -\delta_S/\gamma_S \quad (\gamma_S > 1), \quad \text{or} \quad \delta_M = -(\delta_S/\gamma_S) - 1 \quad (\gamma_S = 1)$$

and the image  $(S \rightarrow M)(\Gamma_S)$  equals  $\Gamma_M^* = (]0, 1[ \times \mathbb{R}) \cup (\{1\} \times [-1, \infty[)$ .

The inverse map is defined on  $\Gamma_M^*$  and satisfies:

$$(M \rightarrow S) : (\beta_M, \delta_M) \mapsto (\gamma_S, \delta_S) \quad \text{with}$$

(100)

$$\gamma_S = 1/\beta_M, \quad \delta_S = -\delta_M/\beta_M \quad (\beta_M < 1) \quad \text{or} \quad \delta_S = -(\delta_M + 1)/\beta_M \quad (\beta_M = 1).$$

We have thus proven the following result, which provides the final answer to our “exhibition”:

**Theorem 49.** *For any pair*

$$(\beta_M, \delta_M) \in (]0, 1[ \times \mathbb{R}) \cup (\{1\} \times [-1, \infty[),$$

*there exists a source  $S$  in the subclass  $\mathcal{DRIL}(\gamma, \delta)$  with the pair*

$$(\gamma, \delta) \in (]1, +\infty[ \times \mathbb{R}) \cup (\{1\} \times ]-\infty, 0])$$

*which admits Shannon weights  $m(n)$  of the form  $m(n) = \Theta(n^{\beta_M} (\log n)^{\delta_M})$ .*

*The pair  $(\gamma, \delta)$  is related to the pair  $(\beta_M, \delta_M)$  via Equation (100)*

**9.4. Remarks on the final result.** There are three remarks :

(a) The limit case  $(\beta_M, \delta_M) = (1, -1)$  is obtained with sources of the subclass  $\mathcal{DRIL}(1, 0)$ , and for instance with the Farey source.

(b) The previous result does not provide a  $\mathcal{DRIL}$  source whose parameters  $(\beta_M, \delta_M)$  satisfy

$$(101) \quad (\beta_M = 1, \delta_M \in ]-1, 0]) \quad \text{or} \quad (\beta_M = 0, \delta_M > 0).$$

The first case would correspond to Shannon weights very close to a linear behaviour  $\Theta(n(\log n)^{\delta_M})$  with  $\delta_M$  negative near 0.

However, in another paper [ACPRV], the authors study the model called Variable Length Memory Chains (VLMC) and exhibit instances of sources whose parameters satisfy (101). These results are not developed here.

(c) We give some hints on the constant that intervenes in Theorem 49. When the system belongs to  $\mathcal{DRIL}(\gamma, \delta)$ , then, from Proposition 48, the sequence  $q_\tau(n)$  has a constant  $K_\tau$  that is expressed in terms of the pair  $(\beta = \beta_M, \delta = \delta_M)$ ,

$$K_\tau = \beta^{\beta+\delta+1} \quad (\beta = 1) \quad K_\tau = \beta^{\beta+\delta} \quad (\beta < 1),$$

and, from Theorem 23, the precise asymptotic for  $\underline{m}_\mu(n)$  holds in terms of the pair  $(\beta = \beta_M, \delta = \delta_M)$

$$\underline{m}_\tau(n) \sim \left\{ \begin{array}{ll} \frac{\mathcal{E}_\nu(\mathcal{B})}{\psi(0)} & (\delta + 1) \beta^{-(\beta+\delta+1)} \quad \frac{n^\beta}{(\log n)^{\delta+1}} \quad (\beta = 1) \\ \frac{\mathcal{E}_\nu(\mathcal{B})}{\psi(0)} & \frac{\beta^{-(\beta+\delta)}}{\Gamma(\beta+1)\Gamma(1-\beta)} \quad \frac{n^\beta}{(\log n)^\delta} \quad (0 \leq \beta < 1) \end{array} \right\}.$$

There are two kinds of constants. the second group of constants (in the second column) are absolute constants, that only depend on the pair  $(\beta, \delta)$ . But there is also (in the first column) the quotient  $\mathcal{E}_\nu(\mathcal{B})/\psi(0)$  that involves the block dynamical system itself. There is thus an important question: how does the block dynamical system of a system of the  $\mathcal{DRIL}(\gamma, \delta)$  Class evolve as a function of the pair  $(\gamma, \delta)$  –namely, the quotient of its entropy with its invariant density–?

### 9.5. Some instances of the $\mathcal{DRIL}$ Class.

*Our emblematic instance: the Farey source.* The inverse branch  $a$  of the Farey map is  $x/(1+x)$  and its derivative equals  $a'(x) = (1+x)^{-2}$ . Then the fonction  $u'(x) = 1 - (1+x)^{-2}$  satisfies  $u'(x) \in [0, 1]$ . Moreover, one has

$$u'(x) = x \cdot V_0(x) \quad \text{with} \quad V_0(x) = \frac{2+x}{(1+x)^2},$$

and the Farey map belongs to the Class  $\mathcal{DRIL}(1, 0) \cap \mathcal{DR}_w$ .

*Other instances.* We also describe in Figure 3 other instances of the class  $\mathcal{DRIL}(\gamma, \delta)$  with  $(\gamma, \delta) \in \Gamma_S$ . We first remark that the branch  $b$  has no influence on the pair  $(\beta_M, \delta_M)$  which defines the asymptotic behaviour of Shannon weights. The branch  $b$  just intervenes via its value  $b'(0)$ , which may be equal to  $-1$  –case  $\mathcal{DRIL}_w$ – or strictly larger than  $-1$  –Case  $\mathcal{DRIL}_s$ – (See Section 3.3). As Lemma 47(ii) suggests, we limit ourselves to (small) positive integer values of  $\delta$ , for instance,  $\delta = 1$  or  $2$  where the computations become simpler, with

$$V_1(x) = A|\log \frac{x}{2}| \text{ and } V_2(x) = A(\log \frac{x}{2})^2,$$

for a precise choice of constants  $A$ , which depends on the pair  $(\gamma, \delta)$  and is described in Lemma 47(i).

Figure 3 deals with five explicit pairs  $(\beta, \delta)$  for which we indicate the corresponding Shannon pairs  $(\beta_M, \delta_M)$ . The first two pairs are relative to the Class  $(\mathcal{DRIL})_s$ , and the branch  $b$  is the linear complete map. The last three pairs are relative to the Class  $(\mathcal{DRIL})_w$ , and we then consider branches  $b$  with  $b'(0) = -1$ . The final case corresponds to the Farey source.

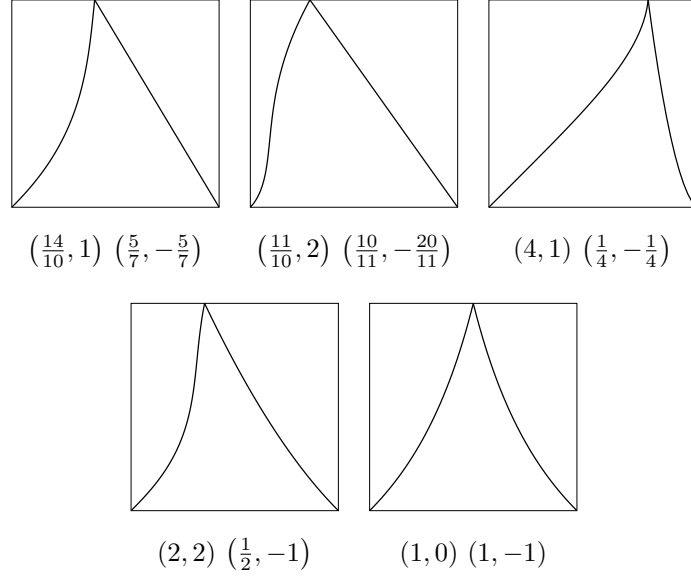


FIGURE 3. Five pairs  $(\gamma, \delta)$ , their corresponding Shannon pairs  $(\beta_M, \delta_M)$ , and their corresponding graphs

**9.6. Application to the Farey source.** The Farey source is described in Section 3.8; it is proven there that the Farey source belongs to the subclass  $\mathcal{DR}_w$  and its block source is defined by the Gauss map. It is proven in Section 9.5 that the Farey source belongs to the subclass  $\mathcal{DRIL}(1, 0)$ . Thus, its block source (the Gauss source) belongs to the Good Class.

Moreover, the main spectral objects of the Gauss map are explicit :

- its stationary density is  $\psi(x) = 1/(\log 2) 1/(1+x)$
- its entropy equals  $\pi^2/(6 \log 2)$

Then we may apply our three main theorems, and obtain the final estimate for the Shannon weights:

**Theorem 50.** *Denote by  $\mu$  a measure with a strictly positive density  $\phi$  of class  $\mathcal{C}^1$ . The Shannon weights  $\underline{m}_\mu(n)$  of the Farey source satisfy*

$$\underline{m}_\mu(n) \sim \phi(0) \frac{\pi^2}{6} \frac{n}{\log n}, \quad (n \rightarrow \infty).$$

As we explain in the next subsection, and to the best of our knowledge, this result is new.

**9.7. Return to Shannon weights.** The notion of coincidence intervals is central in the context of dynamical systems, here defined on the unit interval  $\mathcal{I}$ . We first recall this notion: for any  $x \in \mathcal{I}$  and any depth  $k$ , the interval  $\mathcal{I}_k(x)$  gathers all the reals  $y \in \mathcal{I}$  that belong to the same fundamental  $\mathcal{I}_w$  as  $x$ . Any  $y \in \mathcal{I}_k(x)$  then emits the same prefix of length  $k$  as  $x$ . The random variable  $X_k := \log \mu(\mathcal{I}_k)$  is thus of great interest.

The present paper studies Shannon weights and describes their asymptotic behaviour. Observe that, for  $k \geq 1$ , the function  $x \mapsto \mathcal{I}_k(x)$  is a staircase function which is constant on each interval  $\mathcal{I}_w$  with  $|w| = k$ , and thus Shannon weights exactly coincide with the expectations of the variable  $x \mapsto X_k(x)$ ,

$$\underline{m}_\mu(k) = \mathbb{E}_\mu[X_k], \quad X_k(x) = |\log \mu(\mathcal{I}_k)(x)|.$$

The paper [BCRS23] studies other probabilistic characteristics of the random variable  $X_k := |\log \mu(\mathcal{I}_k)|$ . It introduces in Definitions 2.1 and 2.2 of [BCRS23] two notions of weights (AEW) and (MW) that provide extensions of respective entropies introduced in the paper of Dajani and Fieldsteel [DF01]. It does not consider the notion of Shannon weights.

We now rewrite all these definitions in the ‘‘dynamical’’ context:

**Definition 51.** [BCRS23] *Consider a dynamical source  $\mathcal{S}$ , a measure  $\mu$  on the unit interval, for any depth  $k$ , the random variable  $X_k := |\log \mu(\mathcal{I}_k)|$ , and a real positive sequence  $f : k \mapsto f_k$ .*

(SW)[ $f$ ] *The source  $\mathcal{S}$  has a Shannon weight  $f_k$  if and only if*

$$\mathbb{E}_\mu[X_k] \sim f_k \quad (k \rightarrow \infty)$$

(AEW)[ $f$ ] *The source  $\mathcal{S}$  has a almost everywhere weight  $f_k$  if and only if*

$$X_k(x) \sim f_k \quad [x \text{ almost everywhere}] \quad (k \rightarrow \infty)$$

(MW)[ $f$ ] *The source  $\mathcal{S}$  has a weight (in measure)<sup>12</sup>  $f_k$  if and only if*

$$X_k(x) \sim f_k \quad [\text{in measure}] \quad (k \rightarrow \infty)$$

*In each of the three cases, if the sequence  $k \mapsto f_k$  is linear, of the form  $f_k = c \cdot k$ , the source  $\mathcal{S}$  is said to have entropy  $c$  (resp. Shannon entropy, almost everywhere entropy, entropy in measure).*

The paper [BCRS23] focuses on the notions (AEW) or (MW) and considers many classical sources defined in Number Theory contexts; it studies in particular the Farey dynamical source (called in [BCRS23] the Stern-Brocot source) from these two points of view and proves the following:

- the Farey source satisfies (MW)[ $f$ ] with  $f_k = \phi(0)(\pi^2/6)(k/\log k)$  ;
- the Farey source does not satisfy (AEW).

The two papers (the [BCRS23] paper and the present one) thus study the same sequence

$$Y_k(x) = \frac{X_k(x)}{f_k} - 1 \quad ,$$

(and both in an indirect way). The paper [BCRS23] shows in its Proposition 4.7 that  $Y_k$  converges to 0 in measure, with tools that are quite specific to the Farey source and its close relation to continued fraction expansions (using for instance a deep result due to Khinchin), while the present one obtains the convergence of the

<sup>12</sup>We first recall that a sequence  $X_k$  converges to 0 in measure if and only if

$$\forall \epsilon > 0, \lim_{k \rightarrow \infty} |\{x \in \mathcal{I} \mid |X_k(x)| > \epsilon\}| = 0. \quad (k \rightarrow \infty)$$

expectation  $\mathbb{E}[Y_k]$  to 0, with quite different tools (notably Abelian and Tauberian theorems, together with a renewal theorem) that can be applied to more general contexts.

There is now an important question: Is it possible to directly relate the two results in our setting? There does not exist any general relation between (MW)[ $f$ ] and (SW)[ $f$ ], and one needs extra *sufficient* assumptions on  $Y_k$ : for instance, general domination properties, described in [T11], or a more precise information on the second moment of  $X_k$ , would allow to relate (MW)[ $f$ ] to (SW)[ $f$ ].

Assume, for instance, that the second moment  $\mathbb{E}[X_k^2]$  of  $X_k$  is asymptotically equivalent to the square  $f_k^2 = (\mathbb{E}[X_k])^2$  of the first moment; then the variance  $\text{Var}(X_k)$  would satisfy  $\text{Var}(X_k) = \mathbb{E}[X_k^2] - f_k^2 = o(f_k^2)$ , and Chebyshev's inequality would apply and imply

$$(102) \quad \mu[|Y_k| \geq \varepsilon] = \mu[|X_k - f_k| \geq f_k \varepsilon] \leq \frac{\text{Var}(X_k)}{f_k^2 \varepsilon^2} = o(1),$$

for any fixed  $\varepsilon > 0$ . This would show that  $Y_k$  converges to 0 in measure.

This relation between (MW) and (SW) will be addressed in our setting in a forthcoming paper

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